2. Wess-Zumino Model

Simplest Supersymmetric Model

2.1. Dirac Spinors

- Dirac eq$^{\underline{n}}$: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

  ψ : 4 comp. spinor

- Chiral representation of γ^μ: $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

$$\sigma^\mu = (\mathbb{1}, \sigma^i)$$
$$\bar\sigma^\mu = (\mathbb{1}, -\sigma^i)$$

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R^* \end{pmatrix} \quad , \quad \chi_{L,R} : \text{2-comp. Weyl spinors}$$

$$\psi_L = P_L \psi = \tfrac{1}{2}(\mathbb{1} - \gamma^5)\psi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$$

$$\psi_R = P_R \psi = \tfrac{1}{2}(\mathbb{1} + \gamma^5)\psi = \begin{pmatrix} 0 \\ \chi_R^* \end{pmatrix}$$

$$\bar\psi = (\bar\chi_R, \bar\chi_L) \quad , \quad \bar\psi = \psi^\dagger \gamma^0$$

- Kinetic energy term: $i\bar{\psi}\gamma_\mu \partial^\mu \psi = i\bar{\psi}_L \gamma_\mu \partial^\mu \psi_L + i\bar{\psi}_R \gamma_\mu \partial^\mu \psi_R$

- Mass term: $\bar{\psi}\psi = \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R$

- Charge conj.: $\psi^c = C\bar{\psi}^T = C(\psi^+ \gamma^0)^T = \begin{pmatrix} \chi_R \\ \chi_L^* \end{pmatrix}$, $C = i\gamma^0\gamma^2$

  Note $(\psi_L)^c = \begin{pmatrix} 0 \\ \chi_L^* \end{pmatrix}$ right-handed!
  can replace $q_R \to q_L^c$, $e_R \to e_L^c$

- Majorana spinor: $\psi_M = \psi_M^c \Rightarrow \chi_L = \chi_R$

- Majorana mass term: $\overline{\psi_L^c}\, \psi_L$

FERMION:

$$\psi_L \to \psi_L + \delta\psi_L$$

$$\delta\psi_L = -i\,(\gamma^\mu \mathcal{E})\,\partial_\mu\varphi$$

$$\delta\bar{\psi}_L = i\,(\bar{\mathcal{E}}\gamma^\mu)\,\partial_\mu\varphi^+$$

$$\Rightarrow \delta\mathcal{L}_\psi = \underbrace{-\left(\bar{\mathcal{E}}\gamma^\mu\gamma^\nu\partial_\nu\psi_L\right)\partial_\mu\varphi^+}_{A} + \underbrace{\left(\bar{\psi}_L\gamma^\mu\gamma^\nu \mathcal{E}\right)\partial_\mu\partial_\nu\varphi}_{B}$$

Use: $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]$ ← anti-symm.

$$\Rightarrow \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \partial_\mu\partial^\mu$$

$$A = -\left(\bar{\mathcal{E}}\gamma^\mu\gamma^\nu\partial_\nu\psi_L\right)\partial_\mu\varphi^+$$

integrate by parts $S = \int d^4x\,\mathcal{L}$

$$= \left(\bar{\mathcal{E}}\gamma^\mu\gamma^\nu\psi_L\right)\partial_\nu\partial_\mu\varphi^+ - \partial_\nu\left(\bar{\mathcal{E}}\gamma^\mu\gamma^\nu\psi_L\partial_\mu\varphi^+\right)$$

$$= \bar{\mathcal{E}}\psi_L\,\partial_\mu\partial^\mu\varphi^+ - \partial_\nu\left(\bar{\mathcal{E}}\gamma^\mu\gamma^\nu\psi_L\partial_\mu\varphi^+\right)$$

$$= -\left(\bar{\mathcal{E}}\partial_\mu\psi_L\right)\partial^\mu\varphi^+ + \partial_\nu\left(\bar{\mathcal{E}}\psi_L\partial^\nu\varphi^+ - \bar{\mathcal{E}}\gamma^\mu\gamma^\nu\psi_L\partial_\mu\varphi^+\right)$$

↑ cancels second term in $\delta\mathcal{L}_\varphi$

$$B = \bar{\psi}_L\gamma^\mu\gamma^\nu \mathcal{E}\,\partial_\mu\partial_\nu\varphi = \bar{\psi}_L \mathcal{E}\,\partial_\mu\partial^\mu\varphi$$

$$= \partial_\mu\left(\bar{\psi}_L \mathcal{E}\,\partial^\mu\varphi\right) - \left(\partial_\mu\bar{\psi}_L \mathcal{E}\right)\partial^\mu\varphi$$ ← cancels 1st term in $\delta\mathcal{L}_\varphi$

$$\boxed{\bar{E}Q_M + \bar{Q}_n E}$$

Using the algebra :

or

SCALAR

$$(\delta_{E_1}\delta_{E_2} - \delta_{E_2}\delta_{E_1})\varphi$$

$$= \delta_{E_1}(\bar{E}_2 \psi_L) - \delta_{E_2}(\bar{E}_1 \psi_L)$$

$$= i\left[\bar{E}_1 \gamma^\mu E_2 - \bar{E}_2 \gamma^\mu E_1\right] \partial_\mu \varphi$$

So this does appear to be SUSY.

FERMION?

$$(\delta_{E_1}\delta_{E_2} - \delta_{E_2}\delta_{E_1})\psi = i(\bar{E}_1 \gamma^\mu E_2 - \bar{E}_2 \gamma^\mu E_1)\partial_\mu \psi_L$$
$$+ i E_1^c (\bar{E}_2^c \gamma^\mu \partial_\mu \psi_L) - i E_2^c(\bar{E}_1^c \gamma^\mu \partial_\mu \psi_L)$$

First line is just what we want.

The second line vanishes on-shell: $\gamma^\mu \partial_\mu \psi_L = 0$

Eq^{n} of motion.

Want algebra to be always valid $\rightarrow$ also quantum mech.

i.e. off-shell

$\rightarrow$ Trick: invent extra field F

F: complex scalar field - with NO Kin. term

$$\mathcal{L}_F = F^* F$$

$$\Rightarrow \dim[F] = 2$$

Computation:

$$\left(\delta_{\epsilon_1}\delta_{\epsilon_2}-\delta_{\epsilon_2}\delta_{\epsilon_1}\right)\psi = -i\,(\bar\epsilon_1\,\partial_\mu\psi_L)\gamma^\mu\epsilon_2 + i\,(\bar\epsilon_2\,\partial_\mu\psi_L)\gamma^\mu\epsilon_1$$

Use "Fierz identity":

$$(\bar\epsilon_1\partial_\mu\psi_L)\gamma^\mu\epsilon_2 = -\,\epsilon_1^c(\bar\epsilon_2^c\gamma^\mu\partial_\mu\psi_L$$
$$+\,\partial_\mu\psi_L\,(\bar\epsilon_1\gamma^\mu\epsilon_2)$$

$$= i\,(\bar\epsilon_1\gamma^\mu\epsilon_2-\bar\epsilon_2\gamma^\mu\epsilon_1)\,\partial_\mu\psi_L$$

$$+\,i\;\;\epsilon_1^c(\bar\epsilon_2^c\gamma^\mu\partial_\mu\psi_L) - i\,\epsilon_2^c(\bar\epsilon_1^c\gamma^\mu\partial_\mu\psi_L)$$

$$\mathcal{L}_F = F^* \cdot F$$

$\Rightarrow$ Eq$^{\text{ns}}$ of motion: $\quad \dfrac{\partial \mathcal{L}_F}{\partial(\partial_\mu F)} = 0$

$$\Rightarrow \quad F = F^* = 0 \qquad \text{(on-shell!)}$$

- Now include F is SUSY - transformations:

$$\delta F = i\, \bar{\epsilon}^c \gamma^\mu \partial_\mu \psi_L \,, \quad \delta F^* = -i\, \partial_\mu \bar{\psi}_L \gamma^\mu \epsilon^c$$

multiple of eq$^{\text{n}}$ of motion for ψ

$$\Rightarrow \delta \mathcal{L}_F = i\, (\bar{\epsilon}_c \gamma^\mu \partial_\mu \psi_L) F^* - i\, (\partial_\mu \bar{\psi}_L \gamma^\mu \epsilon^c) F$$

Again: vanishes on-shell, but not off-shell

- Modify $\delta \psi_L$

$$\delta \psi_L = -i (\gamma^\mu \epsilon) \partial_\mu \varphi + \epsilon^c F$$
$$\delta \bar{\psi}_L = i\, (\bar{\epsilon} \gamma^\mu) \partial_\mu \varphi^+ + \bar{\epsilon} F^*$$

$\Rightarrow$ extra term to $\delta \mathcal{L}_\psi$ which just cancels $\delta \mathcal{L}_F$ upto a total derivative.

$$\Rightarrow \quad \mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_\psi + \mathcal{L}_F$$

is SUSY invariant <u>AND</u> algebra closes:

$$(\delta_{\epsilon_1} \delta_{\epsilon_2} - \delta_{\epsilon_2} \delta_{\epsilon_1}) X = i\, (\bar{\epsilon}_2 \gamma^\mu \epsilon_1 - \bar{\epsilon}_1 \gamma^\mu \epsilon_2) \partial_\mu X \,, \quad X = \varphi, \psi, F$$

2.3 Supersymmetry + Interactions

$$\mathcal{L}_{Free} = \partial_\mu \phi_i^+ \partial^\mu \varphi_i + i\, \overline{\psi}_{Li} \gamma^\mu \partial_\mu \psi_{Li} + F_i^* F_i$$

Multiplets: (φ_i, ψ_i, F_i) $\quad i = 1, \dots, N$

- Now include interactions. The most general form which is renorm:

$$\mathcal{L}_{INT} = -\frac{1}{2} W^{ij}\, \overline{\psi^c_{Li}}\, \psi_{Lj} - W^i F_i + \text{c.c.}$$

- Note: $\overline{\psi^c_{Li}}\, \psi_{Lj} = \chi_{Li} \chi_{Lj}$ symmetric in $i \leftrightarrow j$

$\Rightarrow W^{ij}$ symmetric

- $W^{ij} = W^{ij}(\varphi, \varphi^*)$ renormalisability linear in φ, φ^*

$\dim(\overline{\psi^c_L}\, \psi_L) = 3$

- Similarly $W^i(\varphi, \varphi^*)$ quadratic

- Note: $\Rightarrow$ no pure φ, φ^* interaction in $\mathcal{L}$

For example: $\lambda (\varphi\varphi^*)^2$

Since: $(\delta\varphi)\varphi^* (\varphi\varphi^*) = (\varepsilon \psi_L)\, \varphi^* \varphi \varphi^*$

can not be cancelled by anything — think about it.

Now must require

$$\boxed{\delta_{SUSY} \mathcal{L}_{INT} = 0}$$

Since $\delta \mathcal{L}_{FREE} = 0$

COMPUTATION:

(A) PART WITH 4-Spinors

$$\delta \mathcal{L}_{4\text{-Spin}} = -\frac{1}{2} \frac{\delta W^{ij}}{\delta \varphi_k} \delta\varphi_k \, \bar{\Psi}^c_{Li} \Psi_{Lj} - \frac{1}{2} \frac{\delta W^{ij}}{\delta \varphi^+_k} (\delta\varphi^+_k)(\bar{\Psi}^c_{Li} \Psi_{Lj})$$

$$= \underbrace{-\frac{1}{2} \frac{\delta W^{ij}}{\delta \varphi_k} (\bar{E} \Psi_{Lk})(\bar{\Psi}^c_{Li} \Psi_{Lj})}_{1} \underbrace{- \frac{1}{2} \frac{\delta W^{ij}}{\delta \varphi^+_k} (\bar{\Psi}_{Lk} E)(\bar{\Psi}^c_{Li} \Psi_{Lj})}_{2}$$

1 – can not be cancelled by anything in $\delta\mathcal{L}$

but Fierz-identity

$$(\bar{E} \Psi_{Li})(\bar{\Psi}^c_{Lj} \Psi_{Lk}) + (\bar{E} \Psi_{Lj})(\bar{\Psi}^c_{Lk} \Psi_{Li}) + (\bar{E} \Psi_{Lk})(\bar{\Psi}^c_{Li} \Psi_{Lj}) = 0$$

$\Rightarrow$ 1 vanishes if $\frac{\delta W^{ij}}{\delta \varphi_k}$ is symmetric under

$$i \leftrightarrow j, \quad i \leftrightarrow k, \quad j \leftrightarrow k$$

2 No such identity

No such identity $\rightarrow$ $\overline{\psi}_{Lk} \rightarrow \chi_k^*$

$$\Rightarrow \boxed{\frac{\delta W^{ij}}{\delta \varphi_k^\dagger} = 0}$$

very important for Higgs physics

$$W^{ij} = W^{ij}(\varphi)$$

$$\Rightarrow \quad W^{ij} = M^{ij} + y^{ijk}\varphi_k$$

M^{ij}: const. mass matrix – symmetric

y^{ijk}: symmetric Yukawa coupling tensor.

- Can write: $W^{ij} = \frac{\delta^2}{\delta\varphi_i \delta\varphi_j} W$ $\quad \left(\frac{\delta W^{ij}}{\delta \varphi_k} \text{ symm!} \right)$

$$\boxed{W = \frac{1}{2} M^{ij}\varphi_i\varphi_j + \frac{1}{6} y^{ijk}\varphi_i\varphi_j\varphi_k}$$

Superpotential

- analytic function of scalar fields φ_i

$$\rightarrow \delta \mathcal{L}_{int}(\partial) = -i W^{ij}(\partial_\mu \varphi_j)\overline{\psi}_i^c \gamma^\mu \varepsilon - i W^i \partial_\mu \overline{\psi}_i^c \gamma^\mu \varepsilon + c.c.$$

$\uparrow (\delta F_i) W^i$

Since $W^{ij} = \frac{\delta^2}{\delta\phi_i \delta\phi_j} W$

$$\Rightarrow \quad W^{ij} \partial_\mu \varphi_j = \partial_\mu \left(\frac{\delta W}{\delta \varphi_i} \right)$$

$\Rightarrow \quad \delta \mathcal{L}_{int}(\partial)$ is a total derivative iff

$$W^i = \frac{\delta W}{\delta \phi_i} = M^{ij} \varphi_j + \frac{1}{2} y^{ijk} \varphi_j \varphi_k$$

Straightforward to show that remaining terms cancel

What is $\mathcal{L}_{TOT}$?

$$\mathcal{L}_{TOT}(F) = F_i F_i^* \, W^i F_i \, W_i^* F_i^*$$

Eq^n of motion $\Rightarrow$ $F_i = W_i^*$

$$\mathcal{L} = \partial_\mu \varphi_i^* \partial^\mu \varphi_i + i \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L - \frac{1}{2} \left(W^{ij} \bar{\psi}_L^c \psi_L + W^{ij*} \bar{\psi}_L \psi_L^c \right) - W_i W_i^*$$

<u>Explicitly:</u>

$$\mathcal{L} = \partial_\mu \varphi_i^* \partial^\mu \varphi_i + i \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L - \left(\frac{1}{2} M^{ij} \bar{\psi}^c_{Li} \psi_{Lj} + \frac{1}{2} y^{ijk} \varphi_i \bar{\psi}^c_{Lj} \psi_{Lk} + c.c. \right)$$

$$- V(\varphi, \varphi^*)$$

$$V(\varphi, \varphi^*) = W_i W_i^* = F_i F_i^*$$

$$V(\varphi,\varphi^*) = \left(\frac{1}{2} M^*_{ik} M_{kj} \varphi_i^* \varphi_j + \frac{1}{2} M_{in} y^*_{jkn} \varphi_i \varphi_j^* \varphi_k + c.c.\right)$$

$$+ \frac{1}{4} y^{ijn} y^*_{kln} \varphi_i \varphi_j \varphi_k^* \varphi_l^*$$

<u>Note</u> $m^2\varphi^2 \sim m \bar{\psi}_L \psi_L$ — same mass!

2.4. Gauge Interactions: (Brief)

Multiplet: $[\lambda^a (s=\frac{1}{2}),\ A^a_\mu (s=1),\ D^a (s=0)]$

$$\mathcal{L}_{gauge} = +\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i\,\bar{\lambda}^a \gamma^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

$$D_\mu \lambda^a = \partial_\mu \lambda^a - g f^{abc} A^b_\mu \lambda^c$$

With gauge interactions replace

$$\partial_\mu \varphi_i \longrightarrow D_\mu \varphi_i = \partial_\mu \varphi_i + ig A^a_\mu (T^a \varphi)_i$$

$$\partial_\mu \psi_{Li} \longrightarrow D_\mu \psi_{Li} = \partial_\mu \psi_{Li} + ig A^a_\mu (T^a \psi)_i$$

$$\Rightarrow \boxed{\begin{aligned} \mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{(\psi,\varphi)}(D_\mu) - \sqrt{2} g \left[(\varphi^*_i T^a \psi^c_{Li}) \lambda^a + c.c \right] \\ + g (\varphi^* T^a \varphi) D^a \end{aligned}}$$

That is the most general SUSY-invariant Lagrangian!

$\Rightarrow$ Eq^{ns} of motion

$$D^a = -g\,(\varphi_i^* T^a \varphi_i)$$

$$\boxed{\begin{aligned} V(\varphi,\varphi^*) &= F_i^* F_i + \frac{1}{2}\sum_a D^a D^a \\ &= W_i^* W_i + \frac{1}{2}\sum_a g_a^2 (\varphi^* T^a \varphi)^2 \end{aligned}}$$

Scalar potential.

$$W_i = \frac{\delta W}{\delta \varphi_i} = M_{ij}\varphi_j + \frac{1}{2} y_{ijk}\varphi_j\varphi_k$$

We will see: in MSSM only quartic Higgs term

comes from $D^a \cdot D^a$

$\Rightarrow$ quartic Higgs interaction $\sim g^2$

which is known

$$\Longrightarrow \quad \boxed{m_{h^0} \lesssim 135 \text{ GeV}}$$

Best chance to experimentally exclude SUSY.

2.5 Supersymmetry breaking:

- Given $\mathcal{L}_{TOT} \Rightarrow m(e) = m(\tilde{e})$

EITHER SUSY is wrong, or it must be broken.

- WE HAVE HERE CONSIDERED ONLY GLOBAL SUSY

$$E = \begin{pmatrix} 0 \\ \varepsilon_R^* \end{pmatrix} \quad - \quad \text{CONST. SPINOR}$$

- $E = E(x) \Longrightarrow$ LOCAL SUPERSYMMETRY

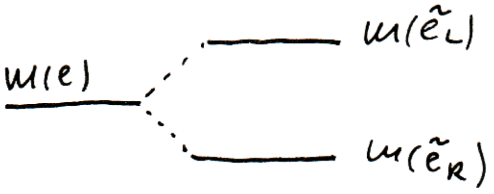
RECALL: $[\bar{E}Q_M, \bar{Q}_M E] = \bar{E}\gamma^\mu E\, P_\mu$

If $E = E(x)$, then this is a local coordinate transformation $\longrightarrow$ gravity

LOCAL SUPERSYMMETRY $\longrightarrow$ <u>SUPERGRAVITY</u>

- SPONTANEOUS BREAKING OF <u>GLOBAL</u> SUSY IS NOT POSSIBLE:

$$\frac{1}{2}\left(m^2(\tilde{e}_L) + m^2(\tilde{e}_R)\right) = m(e)$$



- $\longrightarrow$ MUST GO TO SUPERGRAVITY

$\mathcal{L}_{SUGRA}$ — VERY COMPLICATED

- BOTTOM LINE AFTER SPONTANEOUS BREAKING AT LOW ENERGY (100 GeV)

$$\mathcal{L} = \underbrace{\mathcal{L}_{GLOBAL-SUSY}}_{\text{AS BEFORE}} + \mathcal{L}_{SUSY-BREAK.}$$

- $\mathcal{L}_{SUSY-BREAK} = m_i^2 \varphi_i^+ \varphi_i$ $(m(\tilde{e}_L), m(\tilde{e}_R), m(\tilde{u}_L), \dots)$ scalar mass

$+ m_A \bar{\lambda}^A \lambda^A$ $A = SU(3), SU(2), U(1)$ GAUGINO MASS

$$\left.\begin{array}{l} + A_{ijk}\, y^{ijk} \varphi_i \varphi_j \varphi_k \\ + B_{ij}\, M^{ij} \varphi_i \varphi_j \end{array}\right\} \text{tri- \& bi-linear scalar interactions}$$

- Special feature: these terms preserve solution to hierarchy problem
  $\longrightarrow$ only log. divergences

- No: $M^{ij} \bar{\psi}^c_{Li} \psi_{Lj}$ mass term
  $C^{ijkl} \varphi_i \varphi_j \varphi_k \varphi_l$ quartic scalar interaction

In a large class of SUGRA-models

- $m_i^2 = m_0^2 \;,\; \forall i$ — universal scalar mass
  → at M_{GUT}

$$m(\tilde{e}_L, M_{GUT}) = m(\tilde{q}_L, M_{GUT}) = m(\tilde{\nu}_L, M_{GUT}) = \ldots = m_0$$

- $m_{SU(3)} = m_{SU(2)} = m_{U(1)} = m_{1/2}$ — universal gaugino mass at M_{GUT}

- $\left.\begin{aligned} A_{ijk} &= A \\ B_{ij} &= B \end{aligned}\right\} \forall i,j,k$ universal scalar interactions

⟶ minimal supergravity, (mSUGRA)

Otherwise: more than 100 <u>new</u> parameters in MSSM