Cosmology and structure formation

Simon White

(Script by Thomas Mädler)

6^{th} - 10^{th} October 2004

Contents

1 Preface 1

2 Cosmological principle 1

3 Free particle motion and thermodynamics 1
3.1 Massive nonrelativistic particles . . . 1
3.2 Relativistic particles . . . 2
3.3 False vacuum and cosmological constant . . . 3

4 Angles and luminosities 4

5 Relativistic cosmologies 4

6 Radiation– dominated universe 5

7 Matter dominated universe 6

8 Horizon 7

9 Inflation 8
9.1 Flatness problem . . . 8
9.2 Horizon problem . . . 9
9.3 Monopole problem . . . 9
9.4 Structure cluster . . . 9

10 Thermal history of the universe 10

11 Structure formation 11

1 Preface

This is a copy of Simon White's Lecture held during the Astroteilchenschule 2004 in Obertrubach Bärnfels. The script has been generated by lecture notes of Svenja Klages (MPIK Heidelberg) and Thomas Mädler (Landessternwarte Königstuhl-Heidelberg). Any errors are most probably originated by incorrect lecture notes.

2 Cosmological principle

"Not only the laws of nature but also the events occurring in nature must appear the same in all directions to all observer" Milne 1935

- generalised Copernican principle
- universe is isotropic for all fundamental observers (FO)
- obviously false!
  relevant assumption $\frac{\delta\Phi_{Grav.}}{c^2} << 1$ for all structures
- the only possible motion of FOs are uniform expansions or extractions, Hubble (1926) showed that our universe is expanding
- there exist maximally symmetric 3–surfaces on which $\rho,\ p,\ T$ are constant
- Robertson Walker metric
  $$d\tau^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1-k\,r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right],$$
  where $r,\ \theta,\ \phi$ are comoving coordinates and $a(t)$ is called cosmic expansion factor; they can be determined from General Relativity (GR) and Equation of State (EoS); k is a curvature parameter and can satisfy
  $$k = \begin{cases} +1 & \text{pos. curved 3–system } V = 2\pi a^3 \\ 0 & \text{Euclidian space with spatially infinite volume} \\ -1 & \text{neg.curved space with spatially infinite volume} \end{cases}$$
- the distances are known to be time–like if $d\tau > 0$, null– or light–like if $d\tau = 0$ or space–like in case of $d\tau < 0$

3 Free particle motion and thermodynamics

3.1 Massive nonrelativistic particles

Particles are accelerated w.r.t. their local FO. Hence their *peculiar velocities* satisfies

$$\delta v_p + (v_p\delta t)\,\frac{\dot{a}}{a} = 0 \quad\Rightarrow\quad v_p \propto \frac{1}{a}.$$

this for a non–rel. gas $\rho \propto a^{-3}$, $T \propto a^{-2}$

$$p \propto \rho T \propto \rho^{5/3} \qquad p << \rho c^2$$

More generally $dU = -pdV \ U = \rho V \propto a^3$

$$\frac{\rho}{da} + 3\,\frac{p+\rho}{a} = 0$$

3.2 Relativistic particles

Consider light signal from a FO at r_e propagating to the origin

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} - \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)},$$

where δt_e and δt_0 are emitted and observed spacing of successive wave crests.

$$\rightarrow \frac{\delta t_0}{a(t_0)} = \frac{\delta t_e}{a(t_e)} \quad \Rightarrow \quad \text{period} \propto \lambda \propto a$$

wavelength of all photons are *redshifted* by $\frac{a(t_0)}{a(t_e)}$, take t_0 to be the present time this gives standard redshift

$$1 + z = \frac{a(t_0)}{a(t_e)}$$

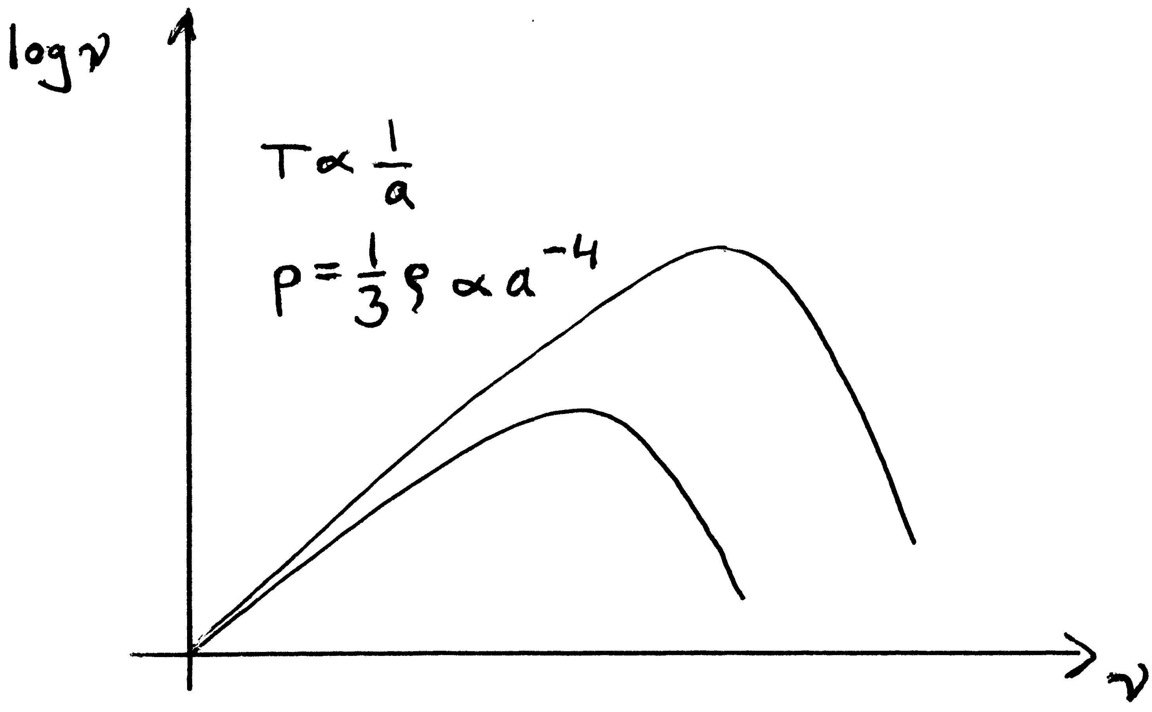
For a photon gas

$$\lambda \propto a \qquad \text{for each photon}$$

$$n_\gamma \propto a^{-3}$$

For a black body

$$n_\gamma \propto T^3 \qquad \lambda_{\text{Peak}} \propto \frac{1}{T}$$



Blackbody radiation remains blackbody radiation as the universe expands.

3.3 False vacuum and cosmological constant

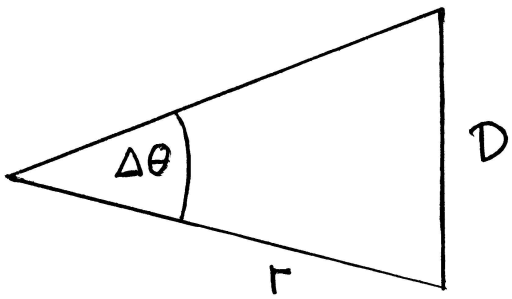
$$\frac{d\rho}{da} + 3\,\frac{p+\rho}{a} = 0$$

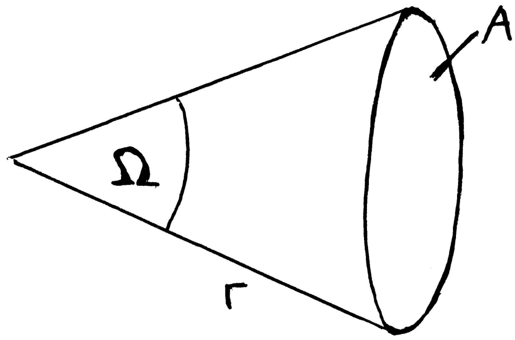
If we have $p = const$ (e.g. false vacuum) $\rightarrow p = -\rho$. The Equation of State gives

non-rel.	rel.	for Cosmological Constant
$p = 0$	$p = \frac{1}{3}\rho$	$p = -\rho$

More generally people write $p = w\rho$ where $w = -1$ in case of a Cosm. Constant. quintessence models have $w > -1$ and are time variable.

4 Angles and luminosities





Consider a source of proper size is D and intrinsic luminosity is L

$$D = a(t)\, r \Delta\Theta = \frac{a(t_0)\, r}{1+z} \Delta\Theta$$

Similarly an area A at the origin subtends at source a solid angle $A = (a(t_0) r)^2\, \Omega$

Received flux/unit are

$$f = \frac{1}{A}\, L\, \frac{\Omega}{4\pi} \left(\frac{a(t)}{a(t_0)}\right)^2 = \frac{L}{4\pi d^2} = \frac{L}{4\pi \left[a(t)\, r_e\, (1+z)\right]^2}$$

One power of $\frac{a(t)}{a(t_0)}$ from photons arrival rate and second from photons energy redshift

$a(t_0)\, r\, (1+z)$ angular size
are known as   distance
$a(t)\, r\, (1+z)$ luminosity

Observed surface brightness is

$$\frac{f}{\pi (\Delta\Theta)^2} = \frac{L}{4\pi^2 D^2}\, (1+z)^{-4}$$

This dimming is independent of the relation between $a(t_0)\, r$ and z

5 Relativistic cosmologies

Einstein Equation

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T^{\lambda}_{\lambda}\right)$$

$R_{\mu\nu}$ is Ricci curvature tensor, $g_{\mu\nu}$ is the metric tensor and $T_{\mu\nu}$ is Energy momentum tensor of matter/isolation. Assume

- RW metric

- isotropic perfect fluids

$$\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} \qquad \text{FRIEDMANN}$$

With an Equation of State and initial conditions this specifies the solution of $a(t)$. Differentiation w.r.t. tie and using 1^{st} law of thermodynamics gives

$$\ddot{a} = -\frac{4\pi}{3}\,G\,(\rho + 3p)\,a$$

Newtonian equation of motion for a uniform, uniformly expanding pressure free sphere of active mass density $\rho + 3p$

$$\text{Equation} \begin{array}{c}\text{accelerates}\\ \\ \text{decelerates}\end{array} \text{for } \rho + 3p \begin{array}{c}< 0\\ \\ > 0\end{array}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,(\rho_{\text{vac}} + \rho_{\text{matter}} + \rho_{\text{rad}}) - \frac{k}{a^2}$$

Define

- $H = \frac{\dot{a}}{a}$ the Hubble parameter $H_0 = 72 \pm 8 \frac{\text{kms}^{-1}}{\text{Mpc}}$
- $\frac{1}{\rho_{\text{crit}}} = \frac{8\pi G}{3H^2}$ critical density $\rho_{\text{crit}} \sim 10^{-29}$ gcm^{-3}
- $\Omega = \frac{\rho}{\rho_{\text{crit}}}$ density parameter for matter, radiation and vacuum which is time dependent

$$\begin{aligned} H^2 &= H^2\Omega_{\text{vac}} + H^2\Omega_{\text{matter}} + H^2\Omega_{\text{rad}} - \frac{k}{a^2} \\ &= H_0^2\Omega_{\text{vac}}(0) + H_0^2\Omega_{\text{matter}}(0)\,(1+z)^3 + H_0^2\Omega_{\text{rad}}(0)\,(1+z)^4 - \frac{k}{a_0}(1+z)^2 \end{aligned}$$

where 0 denotes the present state and $\frac{\dot{a}}{a}$ has been written as a function of $1 + z \equiv \frac{\dot{a}_0}{a} \rightarrow a(t)$

6 Radiation– dominated universe

At high z

$$\begin{aligned} \frac{\dot{a}}{a} &= \frac{8\pi G}{3}\,\rho_\gamma(0)\,\left(\frac{a_0}{a}\right)^4 \\ \frac{a}{a_0} &= \frac{1}{1+z} = \frac{T_0}{T(z)} = \left(\frac{8\pi G\rho_\gamma(0)}{3}\right)^{1/2}\,\sqrt{zt} \end{aligned}$$

Today $\rho_\gamma(0)$ is dominated by CMB $\rho_\gamma = 4.7 \times 10^{-34}$ gcm^{-3} (BB at T=2.71 K
At early times ($z > 10^4$)

$$\frac{T}{10^{10}\ \text{K}} \sim \frac{kT}{1\ \text{MeV}} \sim \frac{\text{Hz}}{10^{10}} \sim \left[\frac{\rho_{\text{tot}}}{10^7\ \text{gcm}^{-3}}\right]^{1/4} \sim \left[\frac{\rho_{\text{Baryon}}}{1\ \text{gcm}^{-3}}\right]^{1/3} \sim \left[\frac{t}{1\ \text{s}}\right]^{-1/2}$$

That is what is called HOT BIG BANG.

7 Matter dominated universe

Measurements of distance–redshift relation

$$
\begin{aligned}
\rightarrow H_0 &= 72 \pm 8 \text{ kms}^{-1}/\text{Mpc} \\
H_0^{-1} &= 14 \text{ Gyr} \qquad \text{age of the universe} \\
\rightarrow \rho_{\text{crit}} &= \frac{3H_0^2}{8\pi G} = 1.0 \times 10^{-29} \text{ gcm}^{-3} \\
\rightarrow \Omega_\gamma(0) &= 5 \times 10^{-5}
\end{aligned}
$$

CMB suggest $\Omega \approx 1$, $\Omega_m(0) = 0.3$, $\Omega_\gamma(0) = 0.7$ and implies therefore a present matter density of $\rho_m(0) = 3 \times 10^{-30}$ gcm^{-3}. The universe switched from from radiation dominated to matter dominated at

$$1 + z \approx \frac{\Omega_m(0)}{\Omega_\gamma(0)} \approx 6000$$

and from matter dominated to vacuum dominated

$$1 + z \approx \frac{\Omega_{\text{vac}}(0)}{\Omega_m(0)} \approx 1$$

- in matter dominated era

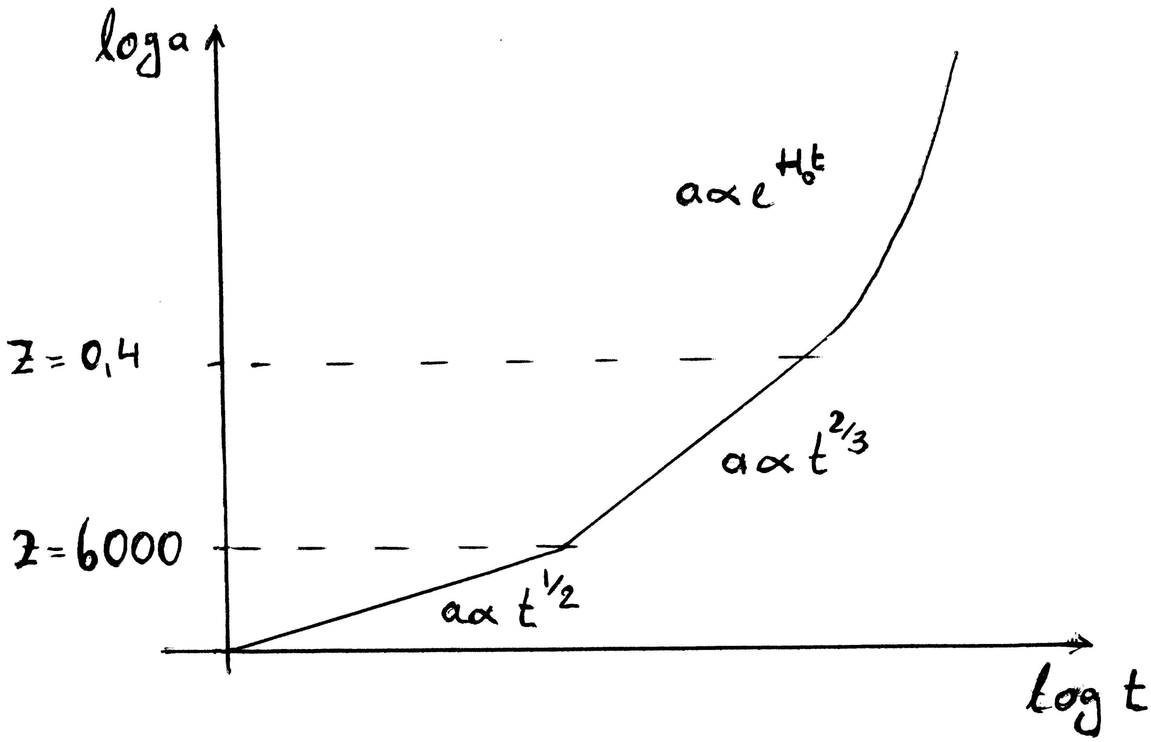
$$
\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G \rho_m(0)}{3} \left(\frac{a_0}{a}\right) = \Omega_m(0)\, H_0^2 \left(\frac{a_0}{a}\right)^3 \\
\frac{a}{a_0} &= \frac{1}{1+z} = \frac{T_0}{T(z)} = \left(\frac{2}{3}\,\Omega_m^{1/2}(0)\, H_0\, t\right)^{2/3} \approx \left(\frac{t}{17 \text{ Gyr}}\right)^{2/3}
\end{aligned}
$$

  This is called Einstein–de Sitter model.

- in vacuum dominated universe

$$
\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G \rho_{\text{vac}}(0)}{3} = \Omega_{\text{vac}}(0)\, H_0^2 = \text{const.} \\
\frac{a}{a_0} &= \frac{1}{1+z} = \exp\left[\Omega_{\text{vac}}^{1/2}\, H_0\, (t - t_0)\right] \approx \exp\left[\frac{t - t_0}{25 \text{ Gyr}}\right]
\end{aligned}
$$

  which is called de Sitter model.



Current value of

$$\rho_{\text{vac}} \approx 7 \times 10^{-30}\ \text{gcm}^{-3}$$

$$\rightarrow T_{\text{equiv}} \sim 30\ \text{K}\ \ \rightarrow kT_{\text{vac}}(0) \sim 0.002\ \text{eV}$$

The 'natural' value for Λ would have

$$kT \sim kT_{\text{Planck}} \approx 10^{19}\ \text{GeV}$$

so the natural value is greater then the observed value by

$$\left(\frac{kT_{\text{Planck}}}{kT_{\text{vac}(0)}}\right)^4 \approx 10^{122} \qquad \textbf{WHY ?}$$

8 Horizon

A light ray emitted by an event at $(r,\ t_1)$ reaches (us) the origin at t_2 where

$$\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_{a_1}^{a_2} \frac{da}{a\sqrt{\frac{8\pi G\rho a^2}{3} - k}}$$

If the t (or a) integral converges as $t_1 \rightarrow 0$ ($a \rightarrow 0$) there exists FO who cannot *yet* have communicated with us. Such observers are beyond our *particle horizon.* Particle horizons expand with t_2.Convergence requires $\rho a^2 \rightarrow \infty$ as $a \rightarrow 0$

- not true for a vacuum dom. universe
- true for rad.dom./ matter dom. universe

9 Inflation

At some early time ($kT \sim kT_{\rm GUT},\ t \sim 10^{-30}$ s) the universe passed through a stage when $\ddot{a} > 0$ ($\leftrightarrow\ \rho a^2$ increases with a) during which a increases by a large factor e.g. passed through a GUT phase transition and get tuck in a phase with $\rho_{\rm vac} > 0$ (and dominant) for a time δt such that

$$H\delta t >> 1 \qquad\qquad H^2 + \frac{8\pi G\rho_{\rm vac}}{3}$$

Inflation ends when transition to new phase is made and vacuum energy is clumped as heat.

$\Rightarrow$ 'Graceful exit problem'

Transition must be completed without building strong inhomogeneities.

9.1 Flatness problem

Universe is already flat $\Omega(0) = \Omega_{\rm vac}(0) + \Omega_m(0) + \Omega_\gamma(0) \approx 1.02 \pm 2\%$ WMAP

$$\begin{aligned} H^2 &= H^2\Omega_{\rm tot} - \frac{k}{a^2} \\ 1 - \Omega_{\rm tot} &= -\frac{k}{H^2a^2} \end{aligned}$$

$$H \propto \begin{matrix} a^{-2} \text{ rad. dom} \\ a^{-3/2} \text{ matt. dom} \end{matrix} \quad \rightarrow \quad 1 - \Omega \propto \left\{ \begin{matrix} a^2 \text{ rad. dom.} \\ a \text{ matter dom.} \\ a^{-2} \text{ vac. dom.} \end{matrix} \right.$$

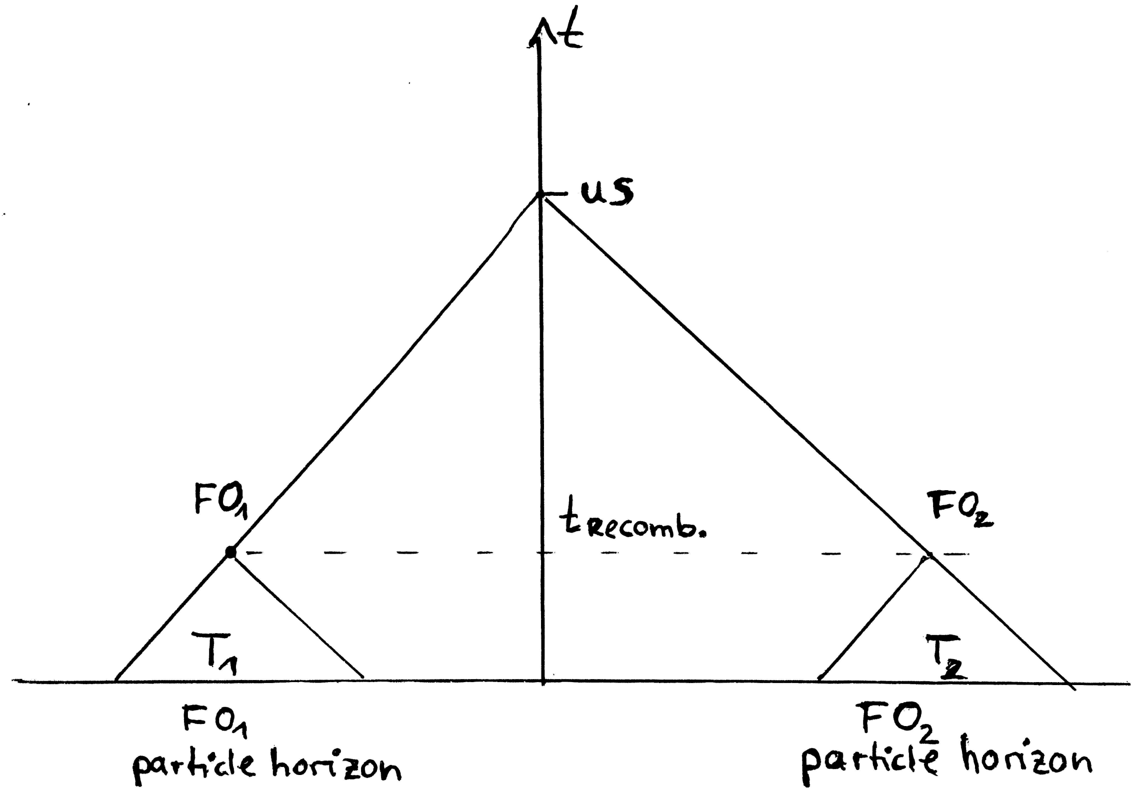
At $t_{\rm Planck}$ $\quad kT_P \sim 10^{19}$ GeV

$$1 - \Omega_{\rm tot}({\rm Pl}) \approx 10^{-58}$$

- Why is $1 - \Omega_{\rm tot}({\rm Pl})$ at $t_{\rm Pl}$ so close to zero?
- Why is the universe so big? (c.f. to $l_p \sim 10^{-33}$ cm)
- Why is the universe so old? (c.f. to $t_p \sim 10^{-43}$ s)

During inflation however $1 - \Omega_{\rm tot} \propto a^{-2}$. So if $H\delta t > 67$ then the expansion is $> e^{67} \sim 10^{29}$. If $1 - \Omega$ is $O(1)$ at the beginning it is $< 10^{-58}$ at the end. If $H\delta t$ *exceeds* 67 then $1 - \Omega \approx 0$ today.

9.2 Horizon problem



T_1 and T_2 are without causal connection, inflation solves this problem

9.3 Monopole problem

Most GUT allow magnetic monopoles with $mc^2 \sim kT_{\mathrm{GUT}}$.
Inflation dilutes their densities by $> 10^{67}$.

9.4 Structure cluster

For a galaxy cluster: mass $10^{15}\, M_\odot \sim 10^{72}\, m_p$

Quantum fluctuations in a non inflating universe cannot give this amplitude $\quad \dfrac{\delta\Phi}{c^2} \sim 10^{-5}$

$$\frac{1}{\sqrt{N}} \sim 10^{-36}$$

$$a \sim e^{Ht} \qquad \rho = \rho_{\mathrm{vac}} = \frac{3\,H^2}{8\pi G} = \mathrm{const}$$

During inflation universe is approximately time invariant

- quantum fluctuation amplitude is time invariant
- fluctuations which "exit" the horizon at different times will have (statistically) the same amplitude (but scales are different)

- scales differ by

$$\frac{\lambda_1}{\lambda_2} = \exp[H\,(t_1 - t_2)]$$

$t \sim \frac{1}{H}$, $\lambda_{horizon} \sim \frac{c}{H}$ At much later time $\lambda\frac{a(t)}{a_0} = ct$ the inflation re–enter the horizon with the same amplitude. If fluctuations originated as zero point fluctuations of inflation field $\rightarrow \frac{\delta\Phi}{c^2} =$ const on all scales as fluctuations cross horizon.
HARRISON/ZEL'DOVICH FLUCTUATIONS

10 Thermal history of the universe

T	t	Process
10^{15} GeV	10^{-36} s	possible epoch of inflation (?); Generation of structure
> 100 MeV	$< 10^{-4}$ s	"quark soup"; no bound hadrons; baryogenesis
~ 100 MeV	10^{-4} s	quark hadron phase transition $\gamma,\ \nu,\ e^-,\ e^+,\ \mu,\ p,\ n$
~ 3 MeV	0.1 s	ν's "decouple" e.g. rate for $e^- + e^+ \leftrightarrow \nu + \bar{\nu}$ becomes long compared to $t \sim 1/H$; universe becomes transparent to ν's.
~ 1 MeV	1 s	rate for $p + e^- \leftrightarrow n + \nu_e$ and $n + e^+ \leftrightarrow p + \bar{\nu}_e$ drops below H, final $\frac{n}{p} = \exp\left(-\frac{\Delta m}{kT}\right) \sim$ 20%
0.3 MeV	10 s	e^+e^- annihilation heats photon gas; establishes final $\frac{n_\gamma}{n_\nu} \sim \frac{11}{3}$ $\frac{n_\gamma}{n_{\rm baryon}} \sim 10^9$
0.1 MeV	100 s	$p + n \rightarrow d$ becomes possible, almost all n are bound to ^{4}He
0.05 Mev	400 s	nucleosynthesis complete 76 % H, 24% He + small amounts of ^{2}H, ^{3}He, ^{7}Li; exact amounts depend strongly on $\frac{n_{\rm baryon}}{n_\gamma}$
3×10^6 K	1 yr	inelastic scattering of γ becomes inefficient Blackbody Spectrum cannot be created *after* this time (gives limit on photon input); observed CMB $\rightarrow$ no sign. input of γ after this time
1000 K	8000 yr	universe switches from radiation to matter dom.
3000 K	3×10^5 yr	Rec. rate $p + e^- \rightarrow H$ dominates re-ionisation by Wien tail of CMB photons: *Recombination*
3 K	13.2 Gyr (today)	transparent to photons

11 Structure formation

$$\rho(\vec{r},\, t) = \bar{\rho}(r)(1 - \delta(\vec{r},\, t)) \qquad \delta << 1$$

linearise the evolution equation to get a linear equation for $\delta(\vec{r},\, t)$.

$$\begin{aligned} \frac{\partial^2 \delta}{\partial t^2} - c_S^2 \nabla^2 \delta &= 0 \\ \frac{d^2 \delta_k}{dt^2} + c_S^2 k^2 \delta_k &= 0 \end{aligned}$$

F.T. the spacial dependence $\delta_k \sim \int d^3x\, \delta(\vec{r},\, t)\, e^{i\vec{k}\cdot\vec{x}}$ to get O.D.E in time for $\delta_{\vec{k}}(t)$. At late times an on long scales

$$\delta_{\vec{k}} = b(t)\delta_{\vec{k}}(t_0) \quad b \text{ independent of } \vec{k};\ b \propto t^{2/3} \propto a$$

Characterise linear fluctuations by their power spectrum

$$\to P(k) \propto k^n \text{ with } n \approx 1$$

Waves with different k have uncorrelated phases therefore δ is a Gaussian random field. Shape of $P(k)$ at recombination depends on

- structure generator
- nature of dark matter