The Cosmological Standard Model and its Parameters

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Geometry and Dynamics

Structure

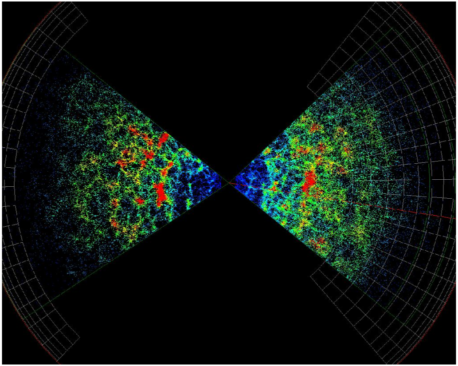
1. Geometry and Dynamics
2. Parameters, Age and Distances
3. Thermal Evolution
4. Recombination and Nucleosynthesis
5. The Growth of Perturbations
6. Statistics and Non-linear Evolution
7. Structures in the Cosmic Microwave Background
8. Cosmological Inflation
9. Dark Energy

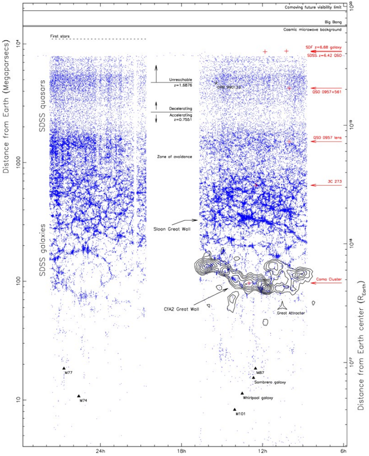
Geometry and Dynamics

Assumptions

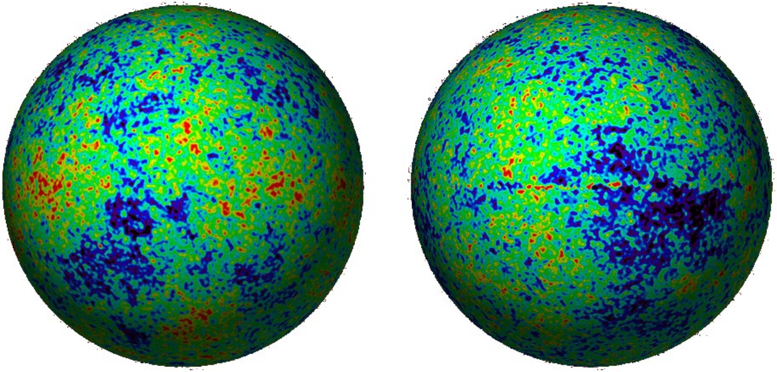
- cosmology rests on two fundamental assumptions:
  1. when averaged over sufficiently large scales, the observable properties of the Universe are isotropic
  2. our position in the Universe is by no means preferred to any other (cosmological principle);
- such a Universe is *homogeneous and isotropic*
- only relevant interaction is gravity, thus we search for cosmological models in General Relativity

Galaxy Distribution





Microwave Background



Metric

- due to symmetry, the 4×4 tensor $g_{\mu\nu}$ has ten independent components: g_{00}, g_{0i}, and g_{ij}; the two fundamental assumptions greatly simplify the metric
- eigentime should equal coordinate time for fundamental observers:

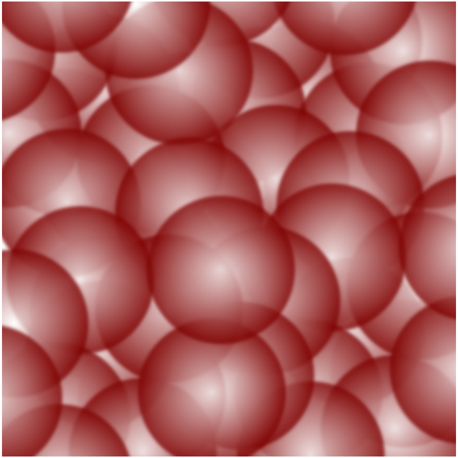
$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 = c^2\mathrm{d}t^2 \;\Rightarrow\; g_{00} = c^2 \tag{1}$$

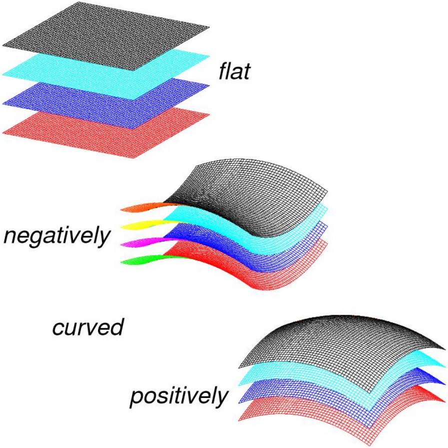
- isotropy requires $g_{0i} = 0$ and spherical symmetry for three-space, thus

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\mathrm{d}w^2 + f_K^2(w)\mathrm{d}\omega^2\right] \text{, with} \tag{2}$$

$$f_K(w) = \begin{cases} K^{-1/2}\sin(K^{1/2}w) & (K > 0) \\ w & (K = 0) \\ |K|^{-1/2}\sinh(|K|^{1/2}w) & (K < 0) \end{cases} \tag{3}$$

Symmetry, Foliation





Redshift

- spatial hypersurfaces can expand or shrink, leading to red- or blueshift
- the propagation condition for light, $\mathrm{d}s = 0$, leads to

$$\frac{\nu_\mathrm{e}}{\nu_\mathrm{o}} = \frac{\lambda_\mathrm{o}}{\lambda_\mathrm{e}} = 1 + \frac{\lambda_\mathrm{o} - \lambda_\mathrm{e}}{\lambda_\mathrm{e}} = 1 + z = \frac{a(t_\mathrm{e})}{a(t_\mathrm{o})} \tag{4}$$

- thus, light is red- or blueshifted by the same amount as the Universe expanded or shrunk between emission and observation

Dynamics

- the dynamics of the metric (2) is reduced to the dynamics of the scale factor $a(t)$; differential equations for $a(t)$ now follow from Einstein's field equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}\,, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3} \qquad (5)$$

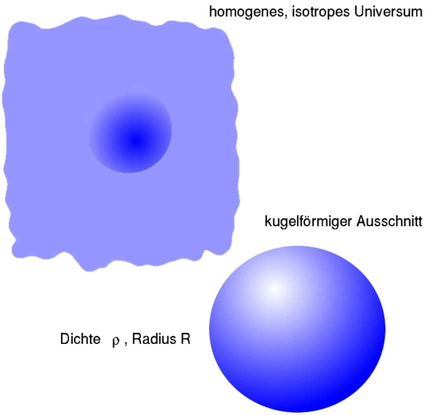
  Friedmann's equations

- the Friedmann equations can be combined to yield the adiabatic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\rho c^2\right) + p\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\right) = 0 \qquad (6)$$

  energy conservation

Remark on Newtonian Dynamics



Remark on Newtonian Dynamics

- (5) can also be derived from Newtonian gravity, except for the Λ term;
- study homogeneous sphere of radius r, ignore surrounding matter; size of the sphere is arbitrary
- pressure term adds to the density because pressure means kinetic energy of particles, which is equivalent to a mass density; yields equation of motion

$$\ddot{r} = -\frac{4\pi G}{3} r \left(\rho + \frac{3p}{c^2}\right) \tag{7}$$

- integrating, using energy conservation, we find

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi G}{3}\rho + \frac{C}{c^2} \tag{8}$$

Parameters, Age and Distances

Parameters, Age and Distances

Forms of Matter

- two forms of matter can broadly be distinguished, relativistic and non-relativistic; they are often called radiation and dust, respectively

- for relativistic bosons and fermions:

$$p = \frac{\rho c^2}{3} \tag{9}$$

for which (6) implies

$$\rho(t) = \rho_0 a^{-4} , \tag{10}$$

($a = 1$ today)

- for non-relativistic matter, $p = 0$ because $p \ll \rho c^2$, and

$$\rho(t) = \rho_0 a^{-3} \tag{11}$$

Parameters (1)

- *Hubble parameter*, relative expansion rate:

$$H(t) \equiv \frac{\dot{a}}{a}\,, \quad H_0 \equiv H(t_0) = 100\,h\,\frac{\mathrm{km}}{\mathrm{s\,Mpc}} = 3.2\times10^{-18}\,h\,\mathrm{s}^{-1} \tag{12}$$

- critical density

$$\rho_{\mathrm{cr}}(t) \equiv \frac{3H^2(t)}{8\pi G}\,, \quad \rho_{\mathrm{cr0}} \equiv \rho_{\mathrm{cr}}(t_0) = \frac{3H_0^2}{8\pi G} = 1.9\times10^{-29}\,h^2\,\mathrm{g\,cm}^{-3} \tag{13}$$

- dimension-less density parameters

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\mathrm{cr}}(t)}\,, \quad \Omega_0 \equiv \frac{\rho(t_0)}{\rho_{\mathrm{cr0}}}\,, \quad \Omega_\Lambda(t) = \frac{\Lambda}{3H^2(t)}\,, \quad \Omega_{\Lambda0} \equiv \frac{\Lambda}{3H_0^2} \tag{14}$$

Parameters (2)

- Friedmann's equation becomes

$$H^2(a) = H_0^2 \left[\Omega_{\mathrm{r}0} a^{-4} + \Omega_{\mathrm{m}0} a^{-3} + \Omega_{\Lambda 0} - \frac{Kc^2}{a^2} \right] \tag{15}$$

specialising to $a = 1$ allows to solve for K,

$$-Kc^2 = 1 - \Omega_{\mathrm{r}0} - \Omega_{\mathrm{m}0} - \Omega_{\Lambda 0} \equiv \Omega_{\mathrm{K}} \tag{16}$$

- final form for Friedmann's equation

$$H^2(a) = H_0^2 \left[\Omega_{\mathrm{r}0} a^{-4} + \Omega_{\mathrm{m}0} a^{-3} + \Omega_{\Lambda 0} + \Omega_{\mathrm{K}} a^{-2} \right] \equiv H_0^2 E^2(a) \tag{17}$$

Parameters (3)

- radiation density exceeded matter density before

$$a_{\mathrm{eq}} = \frac{\Omega_{\mathrm{r0}}}{\Omega_{\mathrm{m0}}} \tag{18}$$

- the density parameters change with time:

$$\begin{aligned} \Omega_{\mathrm{m}}(a) &= \frac{\Omega_{\mathrm{m0}}}{a + \Omega_{\mathrm{m0}}(1-a) + \Omega_{\Lambda 0}(a^3 - a)}\,, \\ \Omega_{\Lambda}(a) &= \frac{\Omega_{\Lambda 0} a^3}{a + \Omega_{\mathrm{m0}}(1-a) + \Omega_{\Lambda 0}(a^3 - a)} \end{aligned} \tag{19}$$

- this implies: $\Omega_{\mathrm{m}}(a) \to 1$ and $\Omega_{\Lambda}(a) \to 0$ for $a \to 0$ regardless of their present values; if $\Omega_{\mathrm{m0}} + \Omega_{\Lambda 0} = 1$, remains valid for $a < 1$

Parameter Values

Hubble constant	h	$0.70^{+0.04}_{-0.03}$ 0.72 ± 0.07	CMB + SDSS HST Key Project
matter density	$\Omega_{\mathrm{m}0}$	0.30 ± 0.04 0.41 ± 0.09	assuming $\Omega_{\mathrm{K}} = 0$ free Ω_{K}
cosmological constant	$\Omega_{\Lambda 0}$	0.70 ± 0.04 0.65 ± 0.08	assuming $\Omega_{\mathrm{K}} = 0$ free Ω_{K}
curvature	Ω_{K}	-0.06 ± 0.04	free Ω_{K}
baryon density	$h^2\Omega_{\mathrm{B}}$ Ω_{B}	0.023 ± 0.001 0.047 ± 0.006	
radiation density	$\Omega_{\mathrm{r}0}$	$(2.494 \pm 0.007) \cdot 10^{-5}$	from CMB temperature
Hubble time	H_0^{-1}	$(1.4 \pm 0.08) \times 10^{10}\,\mathrm{yr}$	
matter-radiation equality	a_{eq}	$(8.3 \pm 1.1) \times 10^{-5}$	

Age and Expansion of the Universe

- since $H = \dot{a}/a$, the age of the Universe is determined by

$$\frac{\mathrm{d}a}{\mathrm{d}t} = H_0 a E(a) \;\Rightarrow\; H_0 t = \int_0^a \frac{\mathrm{d}a'}{a' E(a')} \tag{20}$$

- in a flat universe with $\Omega_{\mathrm{m}0} \neq 0$ and $\Omega_\Lambda = 1 - \Omega_{\mathrm{m}0} \neq 0$:

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{\mathrm{m}0}}} \,\mathrm{arcsinh}\left[\sqrt{\frac{1 - \Omega_{\mathrm{m}0}}{\Omega_{\mathrm{m}0}}}\, a^{3/2}\right] \tag{21}$$

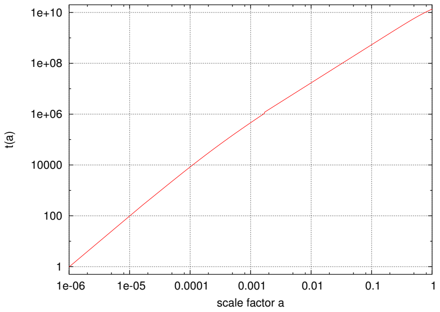
the age of our universe is

$$t(a = 1) = \frac{0.96}{H_0} = 1.35 \times 10^{10}\,\mathrm{yr} \tag{22}$$

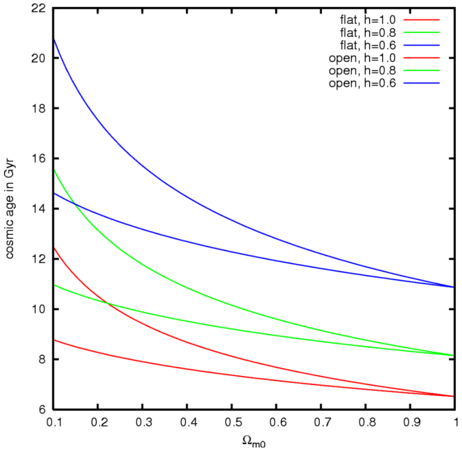
Constraints on the Age of the Universe

- the Universe should be older than its oldest parts
- three ways of measuring the ages:
  1. nuclear cosmo-chronology: decay of long-lived nuclei; $\approx 4.6\,\mathrm{Gyr}$ for the Earth, $7\ldots13\,\mathrm{Gyr}$ for the Galaxy;
  2. ages from stellar evolution: $\gtrsim 13\,\mathrm{Gyr}$ from globular clusters;
  3. cooling of white dwarfs: $\approx 10\,\mathrm{Gyr}$
- $t(a=1) \gtrsim 11\,\mathrm{Gyr}$ needs $H_0 \lesssim 61\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in an Einstein-de Sitter universe

Cosmic Age



1e+10
1e+08
1e+06
10000
100
1
t(a)
1e-06
1e-05
0.0001
0.001
0.01
0.1
1
scale factor a



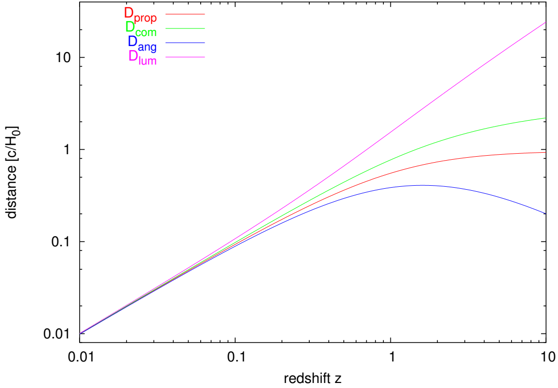
flat, h=1.0
flat, h=0.8
flat, h=0.6
open, h=1.0
open, h=0.8
open, h=0.6
cosmic age in Gyr
22
20
18
16
14
12
10
8
6
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Ω_{m0}

- distance measures are no longer unique in general relativity
- *proper distance* $D_{\rm prop}$, $\mathrm{d}D_{\rm prop} = -c\mathrm{d}t = -c\mathrm{d}a/\dot{a}$
- *comoving distance* $D_{\rm com}$, $\mathrm{d}D_{\rm com} = \mathrm{d}w$
- *angular diameter distance* $D_{\rm ang}$

$$D_{\rm ang}(z_1, z_2) = \left(\frac{\delta A}{\delta\omega}\right)^{1/2} = a(z_2) f_K[w(z_1, z_2)] \qquad (23)$$

- *luminosity distance* $D_{\rm lum}$,

$$D_{\rm lum}(z_1, z_2) = \left[\frac{a(z_1)}{a(z_2)}\right]^2 D_{\rm ang}(z_1, z_2) \qquad (24)$$


D_prop
D_com
D_ang
D_lum
10
1
0.1
0.01
distance [c/H_0]
0.01
0.1
1
10
redshift z

The Hubble Constant

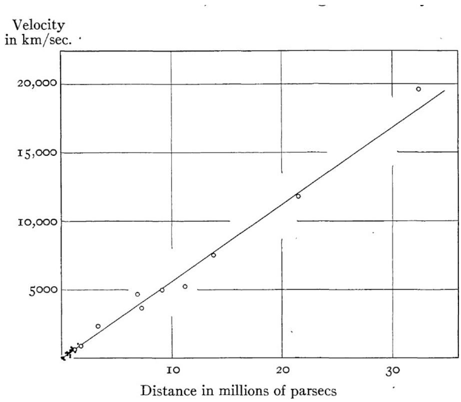
- galaxies move away from us with velocities proportional to their distance; Hubble law,

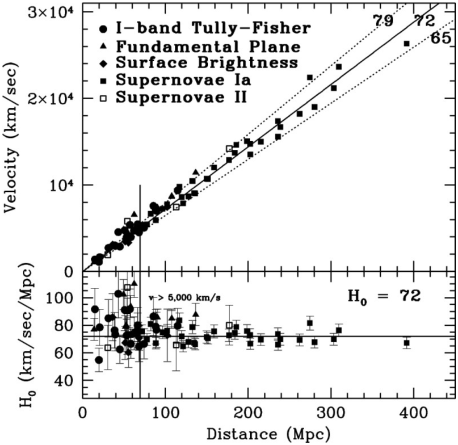
$$D = \frac{cz}{H_0} \quad \Rightarrow \quad v = cz = H_0 D \; ; \tag{25}$$

- local deviations due to peculiar velocities
- accurate distance measurements to distant objects required
- "standard candles": Cepheids, Supernovae, galaxy scaling relations
- result from *Hubble Key Project*:

$$H_0 = 72 \pm 8\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \tag{26}$$

Measurements of the Hubble Constant





Horizons

- between t_1 and $t_2 > t_1$, light can travel across comoving distance

$$\Delta w(t_1, t_2) = \int_{t_1}^{t_2} \frac{c\mathrm{d}t}{a(t)} = c \int_{a(t_1)}^{a(t_2)} \frac{\mathrm{d}a}{a\dot{a}} \propto a^{n/2-1} \quad \text{if} \quad \rho \propto \rho_0 a^{-n} \tag{27}$$

  thus, if $n > 2$, light can only travel by a finite distance; there exists a *particle horizon*

- Hubble radius at a_{eq}, important for structure formation

$$r_{\mathrm{H,eq}} = \frac{c}{H(a_{\mathrm{eq}})} = \frac{c}{H_0} \frac{a_{\mathrm{eq}}^{3/2}}{\sqrt{2\Omega_{\mathrm{m0}}}} \tag{28}$$

Thermal Evolution

1. Assumptions
2. Properties of Ideal Quantum Gases
3. Adiabatic Expansion of Ideal Gases
4. Particle Freeze-Out

Assumptions

- *the universe expands adiabatically* – isotropy requires the universe to expand adiathermally; entropy generation is completely negligible
- *thermal equilibrium can be maintained despite the expansion*
- *the cosmic "fluids" can be treated as ideal gases*
- those assumptions are the starting point of our considerations; they need to be verified as we go along

- for relativistic boson and fermion gases in thermal equilibrium:

$$
\begin{aligned}
n_\mathrm{B} &= 10 g_\mathrm{B} \left(\frac{T}{\mathrm{K}}\right)^3 \mathrm{cm}^{-3} = 1.6 \times 10^{13} g_\mathrm{B} \left(\frac{kT}{\mathrm{eV}}\right)^3 \mathrm{cm}^{-3} \,, \\
n_\mathrm{F} &= \frac{3}{4}\frac{g_\mathrm{F}}{g_\mathrm{B}} n_\mathrm{B} \\
u_\mathrm{B} &= 3.8 \times 10^{-15} g_\mathrm{B} \left(\frac{T}{\mathrm{K}}\right)^4 \frac{\mathrm{erg}}{\mathrm{cm}^3} = 2.35 \times 10^{-3} g_\mathrm{B} \left(\frac{kT}{\mathrm{eV}}\right)^4 \frac{\mathrm{erg}}{\mathrm{cm}^3} \,, \\
u_\mathrm{F} &= \frac{7}{8}\frac{g_\mathrm{F}}{g_\mathrm{B}} u_\mathrm{B} \,, \quad P_\mathrm{B} = \frac{u_\mathrm{B}}{3} \,, \quad P_\mathrm{F} = \frac{u_\mathrm{F}}{3} \\
\frac{s_\mathrm{B}}{k} &= 36 g_\mathrm{B} \left(\frac{T}{\mathrm{K}}\right)^3 \mathrm{cm}^{-3} = 5.7 \times 10^{13} g_\mathrm{B} \left(\frac{kT}{\mathrm{eV}}\right)^3 \mathrm{cm}^{-3} \,, \\
s_\mathrm{F} &= \frac{7}{8}\frac{g_\mathrm{F}}{g_\mathrm{B}} s_\mathrm{B}
\end{aligned}
\tag{29}
$$

Adiabatic Expansion of Ideal Gases

- for relativistic boson or fermion gases in thermal equilibrium

$$P = \frac{u}{3} = \frac{E}{3V} \tag{30}$$

- first law of thermodynamics implies

$$\mathrm{d}E = -P\mathrm{d}V = 3\mathrm{d}(PV) \;\Rightarrow\; P \propto V^{-4/3} \tag{31}$$

  i.e. $\gamma = 4/3$; for non-relativistic ideal gases, $\gamma = 5/3$
- temperature scaling:

$$\begin{aligned} T &\propto P^{1/4} \propto V^{-1/3} \propto a^{-1} \quad \text{(rel.)} \\ T &\propto PV \propto V^{-5/3+1} \propto a^{-2} \quad \text{(non-rel.)} \end{aligned} \tag{32}$$

Particle Freeze-Out

- expansion time-scale during radiation-dominated era

$$t_{\exp} \approx (G\rho)^{-1/2} \propto a^{-2} \qquad (33)$$

- collision rate and time-scale

$$\Gamma \equiv n\langle\sigma v\rangle \propto n \propto T^3 \propto a^{-3} \,, \quad t_{\rm coll} = \Gamma^{-1} \propto a^3 \qquad (34)$$

- ratio $t_{\exp}/t_{\rm coll} \propto a^{-1}$, thermal equilibrium can be maintained despite the expansion at early times; as the universe keeps expanding, thermal equilibrium breaks down when $\Gamma \ll H$
- relativistic particle species retain their thermal-equilibrium density!

Recombination and Nucleosynthesis

1. Neutrino Background
2. Photons and Baryons
3. Recombination Process
4. Nucleosynthesis

The Neutrino Background

- weak interaction

$$\nu + \bar{\nu} \leftrightarrow e^{+} + e^{-} \tag{35}$$

  freezes out when temperature drops to $T_\nu \approx 10^{10.5}\,\mathrm{K} \approx 2.7\,\mathrm{MeV}$
- electron-positron pairs annihilate when temperature drops below $T \approx 2m_\mathrm{e}c^2 \approx 1\,\mathrm{MeV} \approx 10^{10}\,\mathrm{K}$ their decay heats the photon gas, but not the neutrinos
- photon temperature is $\approx 40\%$ higher than neutrino temperature:

$$T_\gamma = \left(\frac{11}{4}\right)^{1/3} T_\nu \tag{36}$$

Photons and Baryons

- number density of baryons today is

$$\begin{aligned} n_{\rm B} &= \frac{\rho_{\rm B}}{m_{\rm p}} = \frac{\Omega_{\rm B}}{m_{\rm p}}\frac{3H_0^2}{8\pi G} = 1.1\times10^{-5}\,\Omega_{\rm B}\,h^2\,{\rm cm}^3 \\ \Omega_{\rm B}h^2 &\approx 0.02 \end{aligned} \tag{37}$$

- the photon number density today is

$$n_\gamma = 407\,{\rm cm}^{-3} \tag{38}$$

- their ratio is constant; about one billion photons per baryon!

$$\eta \equiv \frac{n_{\rm B}}{n_\gamma} = 2.7\times10^{-8}\,\Omega_{\rm B}h^2 \tag{39}$$

The Recombination Process

- approximation: Saha's equation; ionisation fraction x is

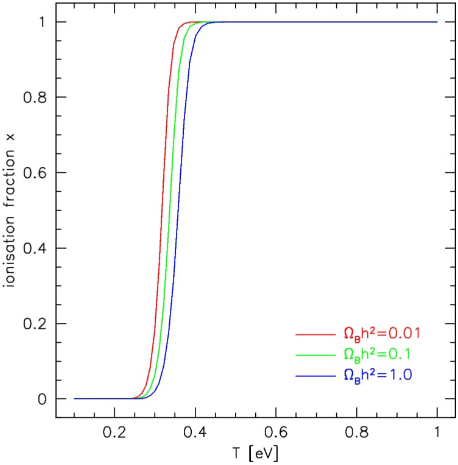
$$\frac{x^2}{1-x} = \frac{\sqrt{\pi}}{4\sqrt{2}\zeta(3)\eta}\left(\frac{m_\mathrm{e}c^2}{kT}\right)^{3/2} \mathrm{e}^{-\chi/kT} \approx \frac{0.26}{\eta}\left(\frac{m_\mathrm{e}c^2}{kT}\right)^{3/2} \mathrm{e}^{-\chi/kT} \tag{40}$$

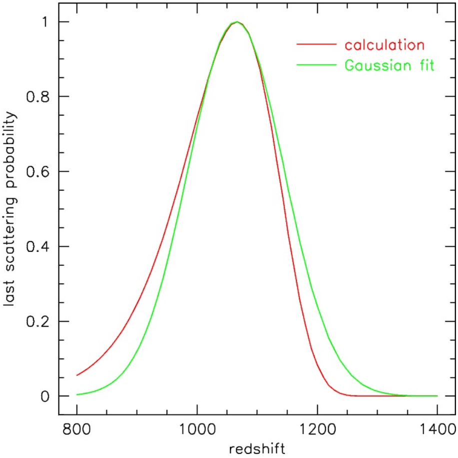
- for recombination to be considered finished, $x \ll 1$ and $x^2/(1-x) \approx x^2$; since $1/\eta$ is a huge number, $kT \ll \chi$ is required for x to be small; for example, putting $x = 0.1$ yields $kT_\mathrm{rec} = 0.3\,\mathrm{eV}$, or

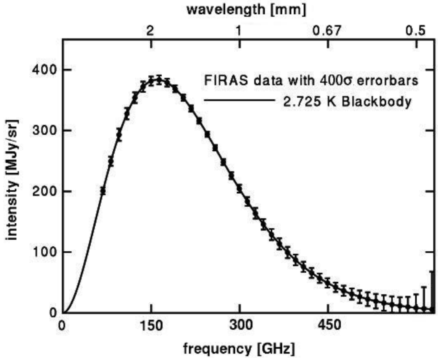
$$T_\mathrm{rec} \approx 3500\,\mathrm{K} \tag{41}$$

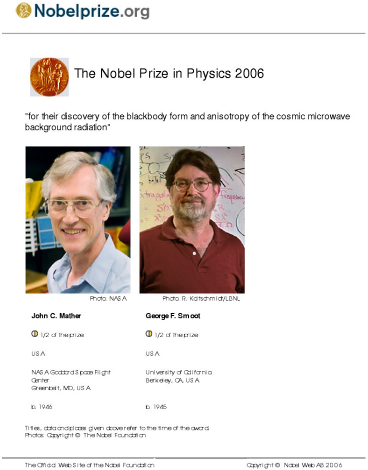
- since $\chi = 13.6\,\mathrm{eV}$, one would naively expect $T_\mathrm{rec} \approx 10^5\,\mathrm{K}$; the very large photon-to-baryon ratio $1/\eta$ delays recombination considerably

Recombination








Nobelprize.org

The Nobel Prize in Physics 2006

"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"

John C. Mather	George F. Smoot
Photo: NASA	Photo: R. Kaltschmidt/LBNL
1/2 of the prize	1/2 of the prize
USA	USA
NASA Goddard Space Flight Center Greenbelt, MD, USA	University of California Berkeley, CA, USA
b. 1946	b. 1945

Titles, data and places given above refer to the time of the award.
Photos: Copyright © The Nobel Foundation

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Nucleosynthesis

- protons and neutrons form when $kT \approx 1\,\mathrm{GeV}$; remain in equilibrium until weak interactions freeze out at $kT \approx 800\,\mathrm{keV}$; baryon-to-photon ratio η is the only relevant parameter,

$$\eta = 10^{10}\eta_{10}\;, \quad \eta_{10} = 273\Omega_\mathrm{B} h^2 \tag{42}$$

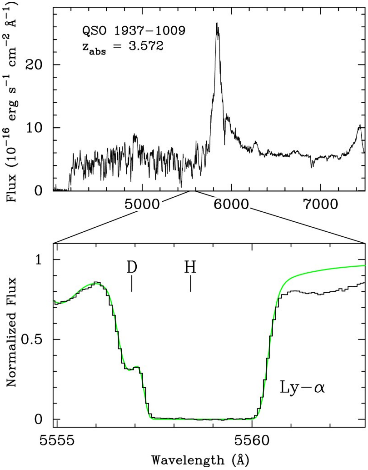
- further fusion builds upon two-body processes, e.g.

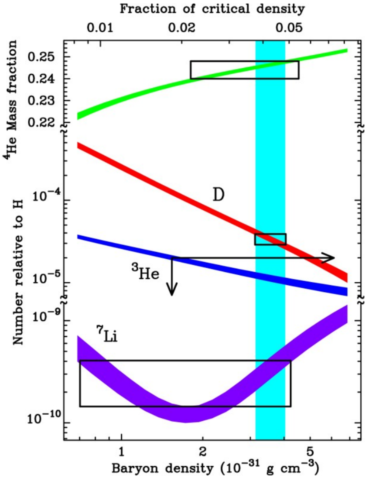
$$\begin{aligned} n + p \to {}^2\mathrm{H} + \gamma\;,\; {}^2\mathrm{H} + {}^2\mathrm{H} \to {}^3\mathrm{He} + n\;, \\ {}^3\mathrm{He} + {}^2\mathrm{H} \to {}^4\mathrm{He} + p\;,\; {}^4\mathrm{H} + {}^3\mathrm{H} \to {}^7\mathrm{Li} + \gamma \end{aligned} \tag{43}$$

- deuterium is crucial! (Gamow criterion)
- the baryon density implied by Big-Bang nucleosynthesis is

$$\Omega_\mathrm{B} h^2 = 0.019 \pm 0.0024 \tag{44}$$

Light-Element Abundances





Chapter V

The Growth of Perturbations

The Growth of Perturbations

1. Newtonian Equations
2. Perturbation Equations
3. Density Perturbations
4. Velocity Perturbations

Newtonian Equations

- continuity equation (mass conservation)

$$\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v}) = 0 \tag{45}$$

- Euler's equation (momentum conservation)

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} + \vec{\nabla}\Phi \tag{46}$$

- Poisson equation

$$\nabla^2\Phi = 4\pi G\rho \tag{47}$$

Perturbation Equations

- decompose density and velocity,

$$\rho(t,\vec{x}) = \rho_0(t) + \delta\rho(t,\vec{x}) \;, \quad \vec{v}(t,\vec{x}) = \vec{v}_0(t) + \delta\vec{v}(t,\vec{x}) \tag{48}$$

  Hubble flow, peculiar velocity:

$$\vec{v} = \dot{\vec{r}} = \dot{a}\vec{x} + a\dot{\vec{x}} = H\vec{r} + a\dot{\vec{x}} = \vec{v}_0 + \delta\vec{v} \tag{49}$$

- comoving coordinates $\vec{x} = \vec{r}/a$, comoving peculiar velocities $\vec{u} \equiv \delta\vec{v}/a$, density contrast $\delta = \delta\rho/\rho_0$
- we are now left with the three equations

$$\dot{\delta} + \vec{\nabla}\cdot\vec{u} = 0 \;, \quad \dot{\vec{u}} + H\vec{u} = -\frac{\vec{\nabla}\delta p}{a^2\rho_0} + \frac{\vec{\nabla}\delta\Phi}{a^2} \;, \quad \nabla^2\delta\Phi = 4\pi G\rho_0 a^2\delta \tag{50}$$

Density Perturbations

- combining these equations, decomposing δ into plane waves yields

$$\ddot{\delta} + 2H\dot{\delta} = \delta\left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right) \tag{51}$$

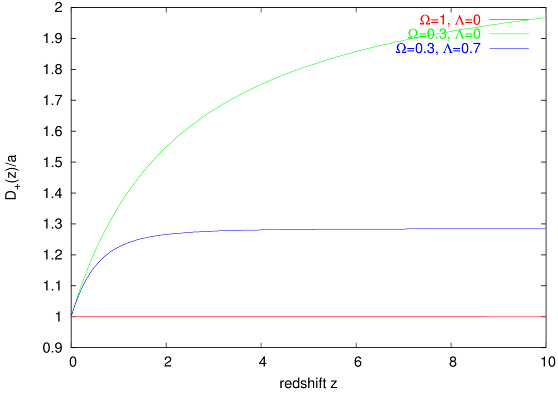
sound speed $c_s^2 = \delta p/\delta\rho$

- assume large perturbations,

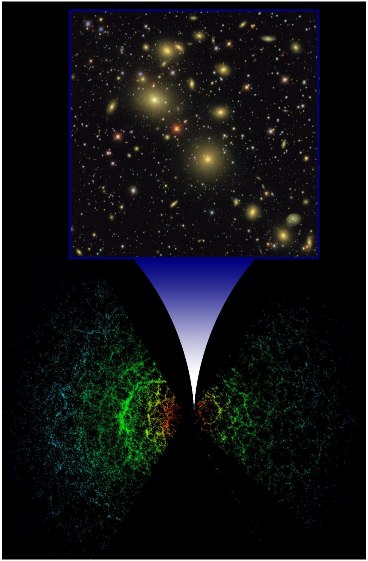
$$\frac{c_s^2 k^2}{a^2} \ll 4\pi G\rho_0 \quad \Rightarrow \quad k \ll a\frac{\sqrt{4\pi G\rho_0}}{c_s} \tag{52}$$

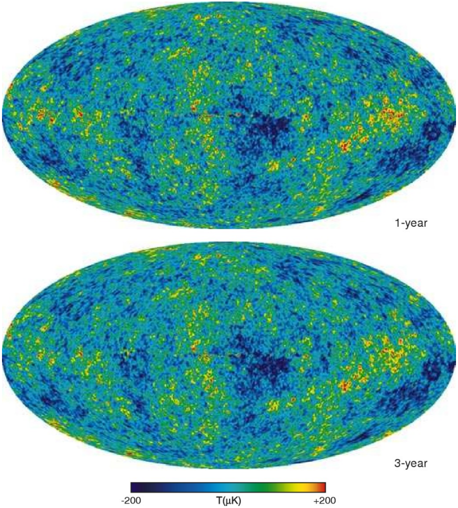
- linear growth factor

$$\delta(a) = \delta_0 D_+(a) \tag{53}$$

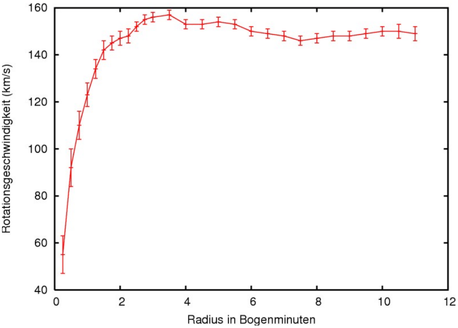

Ω=1, Λ=0
Ω=0.3, Λ=0
Ω=0.3, Λ=0.7
D+(z)/a
redshift z
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0
2
4
6
8
10

Structure Growth and Dark Matter





The Amount of Dark Matter



Velocity Perturbations

- ignoring pressure gradients, the second equation (50) says

$$\dot{\vec{u}} + H\vec{u} = \frac{\vec{\nabla}\delta\Phi}{a^2} \tag{54}$$

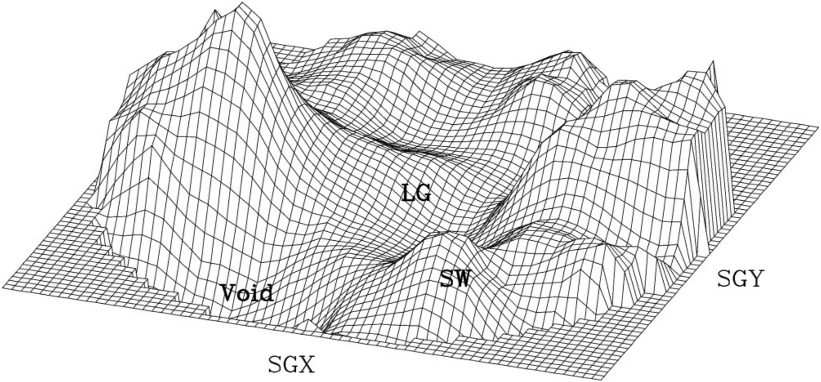
- defining

$$f(\Omega) \equiv \frac{\mathrm{d}\ln D_+(a)}{\mathrm{d}\ln a} \approx \Omega^{0.6} \;, \tag{55}$$

- the peculiar velocity field can be written as

$$\delta\vec{v} = a\vec{u} = \frac{2f(\Omega)}{3aH\Omega}\vec{\nabla}\delta\Phi \tag{56}$$

The Local Cosmic Neighbourhood



Statistics and Non-Linear Evolution

Statistics and Non-Linear Evolution

Power Spectra

- variance of δ *in Fourier space* defines the power spectrum $P(k)$,

$$\langle\hat{\delta}(\vec{k})\hat{\delta}^*(\vec{k}')\rangle \equiv (2\pi)^3 P(k)\delta_\mathrm{D}(\vec{k}-\vec{k}') \quad (57)$$

- variance of δ in real space depends on scale:

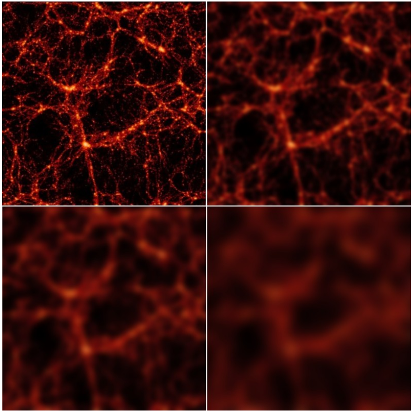
$$\bar{\delta}(\vec{x}) \equiv \int \mathrm{d}^3y\,\delta(\vec{x})W_R(|\vec{x}-\vec{y}|) \quad (58)$$

- the variance of the filtered density-contrast field is

$$\sigma_R^2 = 4\pi \int \frac{k^2\mathrm{d}k}{(2\pi)^3} P(k)\hat{W}_R^2(k) \quad (59)$$

σ_8 is often used for normalising the power spectrum

Smoothing



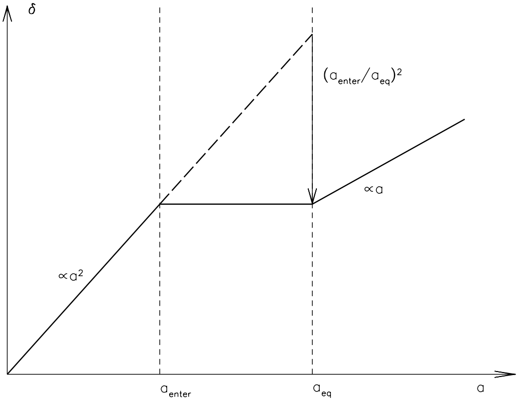
Evolution of the Power Spectrum

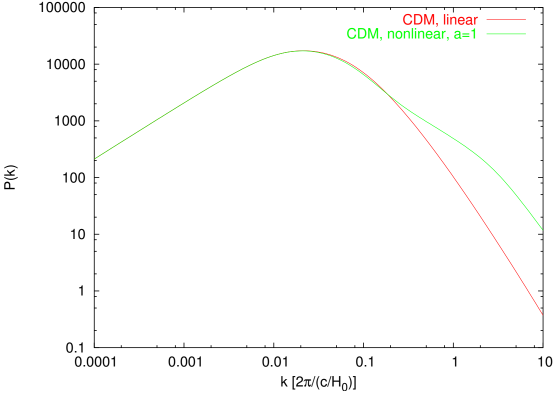
- modes entering the horizon (Hubble radius) while radiation dominates are relatively suppressed compared to larger modes which enter the horizon afterwards
- assuming that the fluctuation power entering the horizon should not depend on time, and working out the suppression for $k > k_0$ yields

$$P(k) \propto \begin{cases} k & (k < k_0) \\ k^{-3} & (k \gg k_0) \end{cases} \tag{60}$$

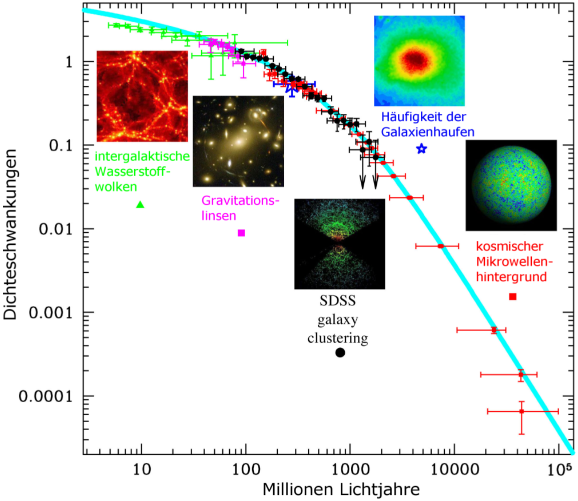
this is the shape of the spectrum for cold dark matter (CDM); hot dark matter (HDM) cuts off the spectrum exponentially

Growth Suppression




CDM, linear
CDM, nonlinear, a=1
P(k)
100000
10000
1000
100
10
1
0.1
0.0001
0.001
0.01
0.1
1
10
k [2π/(c/H0)]

The Observed Power Spectrum



The Zel'dovich Approximation

- a kinematical treatment for following density evolution into the non-linear regime was invented by Zel'dovich; particle trajectories are approximated by

$$\vec{r} = a\left[\vec{x} + \frac{\vec{u}}{Hf(\Omega)}\right] \,, \quad F_{ij} \equiv \frac{\partial r_i}{\partial x_j} \tag{61}$$

- important consequence is the probability distribution $p(\lambda_1, \lambda_2, \lambda_3)$ for the eigenvalues of the deformation tensor F_{ij}:

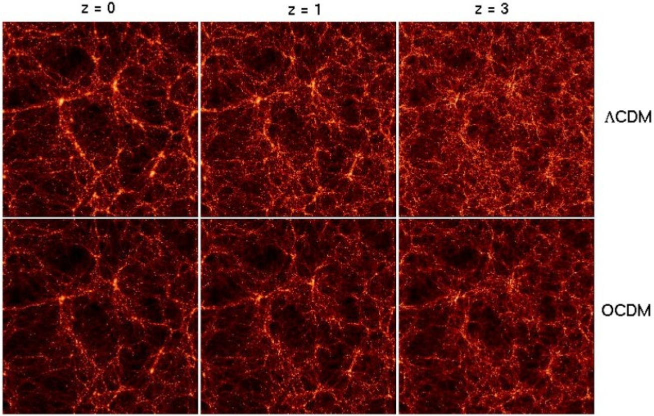
$$p(\lambda_1, \lambda_2, \lambda_3) \propto |(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)| \tag{62}$$

probability for two eigenvalues of F_{ij} to be equal is zero, implying anisotropic collapse! (starting from Gaussian random field)

Nonlinear Evolution

- correct treatment of non-linear evolution requires numerical simulations; decompose the matter distribution into particles whose equations of motion are solved
- non-linear evolution causes mode coupling: while modes of different wave lengths evolve independently during linear evolution, mode coupling in the non-linear evolution moves power from large to small scales as structures collapse
- even if the original density perturbation field δ is Gaussian, it must develop non-Gaussianities during non-linear evolution
- typical behaviour seen in numerical simulations shows the formation of “pancakes”, filaments and voids

Simulations



Structures in the Cosmic Microwave Background

1. Simplified Theory of CMB Temperature Fluctuations
2. CMB Power Spectra and Cosmological Parameters
3. Foregrounds

Simplified Theory: Dipole, Fluctuation Level

- Earth's motion causes temperature dipole,

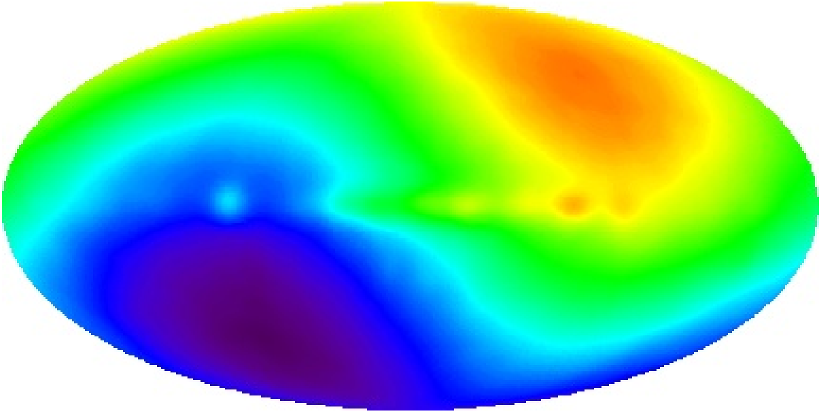
$$T(\theta) = T_0\left(1 + \frac{v}{c}\cos\theta\right) + \mathcal{O}\left(\frac{v^2}{c^2}\right) \tag{63}$$

- $\delta \gtrsim 1$ today implies

$$\delta(a_{\rm CMB}) = \frac{\delta(a=1)}{D_+(a_{\rm CMB})} \gtrsim a_{\rm CMB}^{-1} \approx 10^{-3} \tag{64}$$

  and similar temperature fluctuations in the CMB
- such fluctuations are not found; based on the assumption of weakly interacting dark matter, temperature fluctuations are expected to be $\delta T/T \approx 10^{-5}$: detected by COBE in 1992

Dipole



Simplified Theory: Sachs-Wolfe Effect, Acoustic Oscillations

- perturbation equation for relative temperature fluctuation $\Theta \equiv \delta T / T_0$:

$$\ddot{\hat{\Theta}} + \frac{c^2 k^2}{3}\hat{\Theta} - \frac{k^2}{3}\delta\hat{\Phi} - \frac{\delta\ddot{\hat{\Phi}}}{c^2} = 0 \quad (65)$$

- for small k: Sachs-Wolfe-effect, $\hat{\Theta} \propto \delta\hat{\Phi}/c^2$
- otherwise, oscillator equation for $\hat{\Theta} - \delta\hat{\Phi}/c^2 \equiv \hat{\theta}$; solution assuming $\dot{\hat{\theta}} = 0$ at $t = 0$

$$\hat{\theta}(t_\mathrm{rec}) = \hat{\theta}(0)\cos\left[\frac{ck}{\sqrt{3}}t_\mathrm{rec}\right] \quad (66)$$

$c/\sqrt{3}\,t_\mathrm{rec} \equiv r_\mathrm{s}$: sound horizon; oscillations for $k > 2\pi/r_\mathrm{s}$

Simplified Theory: Silk Damping

- damping occurs due to photon diffusion; diffusion scale:

$$\lambda_\mathrm{D} = \sqrt{N}\lambda\,, \quad \lambda = \frac{1}{n_\mathrm{e}\sigma_\mathrm{T}} \tag{67}$$

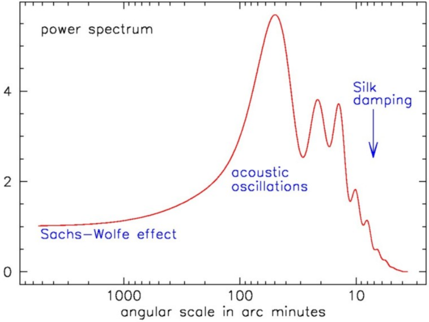
- number of collisions per unit time is $\mathrm{d}N = n_\mathrm{e}\sigma_\mathrm{T}c\mathrm{d}t$; thus,

$$\lambda_\mathrm{D}^2 = \int_0^{t_\mathrm{rec}} \frac{c\mathrm{d}t}{n_\mathrm{e}\sigma_\mathrm{T}} \tag{68}$$

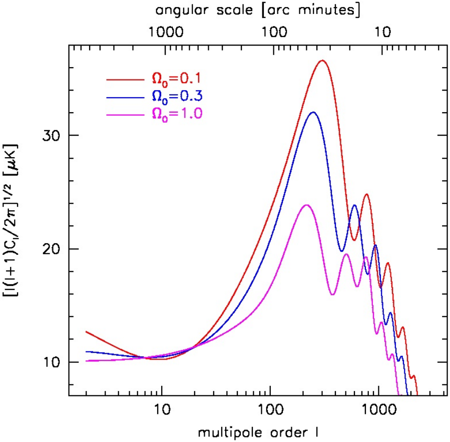
- structures smaller than the diffusion length are damped, hence damping sets in for wave numbers

$$k > k_\mathrm{D} = \frac{2\pi}{\lambda_\mathrm{D}} \tag{69}$$

CMB Spectra



power spectrum
Silk damping
acoustic oscillations
Sachs–Wolfe effect
0
2
4
1000
100
10
angular scale in arc minutes

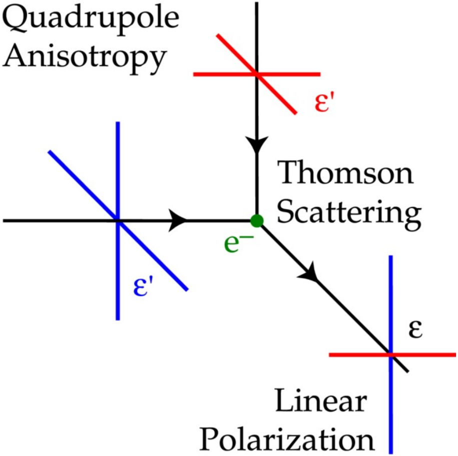


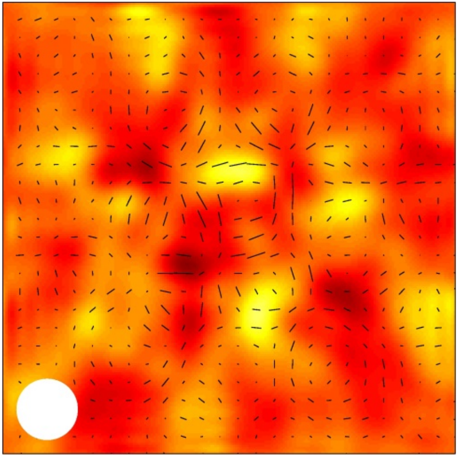
angular scale [arc minutes]
1000
100
10
Ω0=0.1
Ω0=0.3
Ω0=1.0
[l(l+1)Cl/2π]^1/2 [μK]
10
20
30
10
100
1000
multipole order l

Simplified Theory: Polarisation

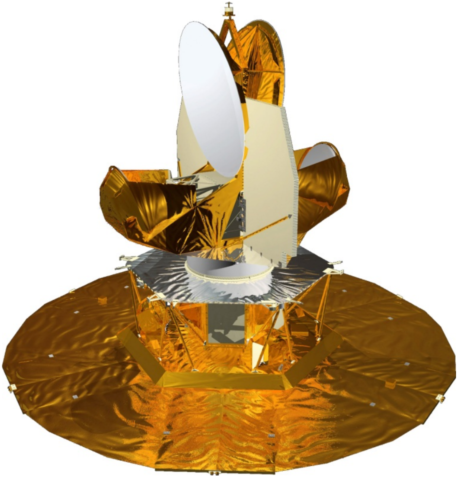
- Thomson scattering is anisotropic; its differential cross section is
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{3\sigma_\mathrm{T}}{8\pi}\left|\vec{e}' \cdot \vec{e}\right|^2 \tag{70}$$
$\vec{e}'$ and $\vec{e}$ are the unit vectors of the incoming and outgoing electric fields
- if infalling radiation has quadrupolar intensity anisotropy, scattered radiation is polarised because it has different intensities in its two orthogonal polarisation directions
- CMB is expected to be linearly polarised to some degree; the intensity of the polarised light should be of order 10% that of the unpolarised light, i.e. it should have an amplitude of order $10^{-6}\,\mathrm{K}$

Polarisation

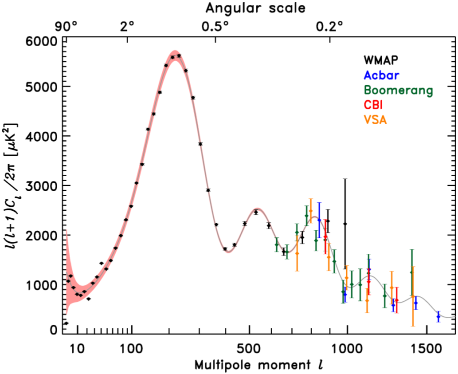


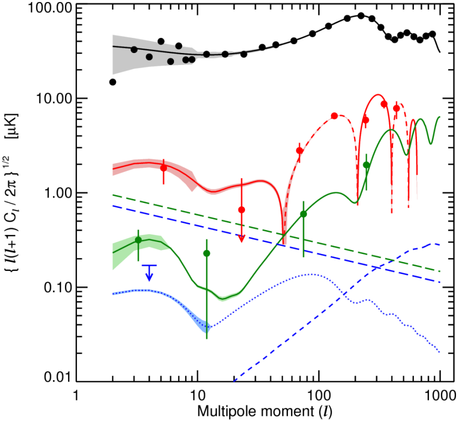


Instruments



Measurements





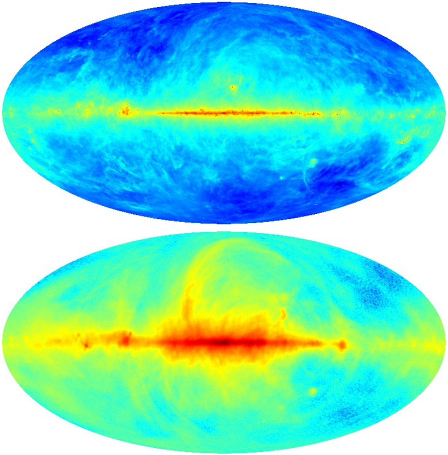
CMB Power Spectra and Cosmological Parameters

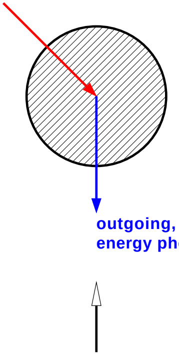
CMB temperature	T_{CMB}	$2.726 \pm 0.002\,\mathrm{K}$
total density	Ω_{tot}	1.02 ± 0.02
matter density	Ω_{m}	0.27 ± 0.04
baryon density	Ω_{b}	0.044 ± 0.004
Hubble constant	h	$0.71^{+0.04}_{-0.03}$
baryon-to-photon ratio	η	$6.1^{+0.3}_{-0.2} \times 10^{-10}$
fluctuation amplitude	σ_8	0.84 ± 0.04
scalar spectral index	n_{s}	0.93 ± 0.03
decoupling redshift	z_{dec}	1089 ± 1
age of the Universe	t_0	$13.7 \pm 0.2\,\mathrm{Gyr}$
age at decoupling	t_{dec}	$379^{+8}_{-7}\,\mathrm{kyr}$
reionisation redshift (95% c.l.)	z_{r}	20^{+10}_{-9}
reionisation optical depth	τ	0.17 ± 0.04

Foregrounds

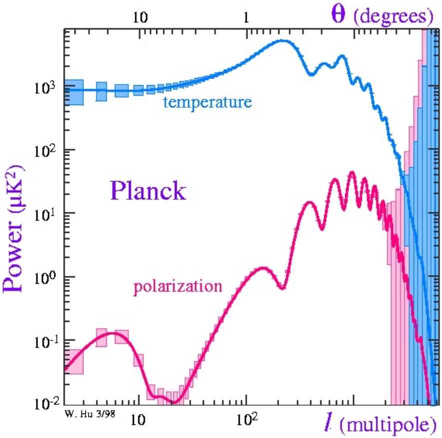
- CMB shines through the entire visible universe on its way to us
- microwave emission from our own Galaxy: warm dust in the plane of the Milky Way with a temperature near $20\,\mathrm{K}$; synchrotron emission from electrons gyrating in the Galactic magnetic field; thermal *bremsstrahlung* from ionised hydrogen; line emission from molecules like CO
- hot plasma in galaxy clusters inverse-Compton scatters microwave background photons to higher energies: Sunyaev-Zel'dovich effect
- point source appearing in the microwave background, such as high-redshift galaxies, planets, asteroids, possibly comets in the Solar System, dust in the plane of the Solar System

Foreground Components





Planck



θ (degrees)
10
1
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
Power (μK²)
temperature
Planck
polarization
W. Hu 3/98
10
10^2
l (multipole)

Cosmological Inflation

1. Problems
2. Inflation

Problems

- angular size of the particle horizon at recombination is

$$\theta_{\rm rec} = \frac{a_{\rm rec}\Delta w(0, a_{\rm rec})}{D_{\rm ang}(0, z_{\rm rec})} \approx \sqrt{\Omega_0 a_{\rm rec}} \approx 1.7^\circ \sqrt{\Omega_0} \tag{71}$$

  causal connection? *horizon problem*

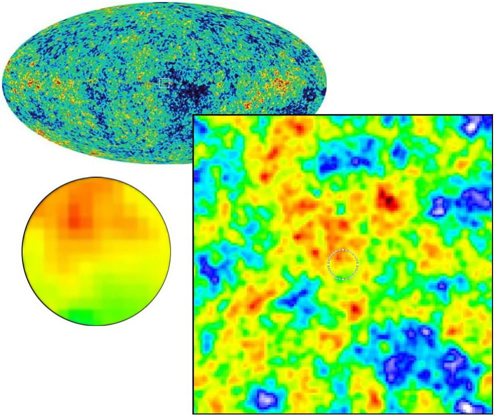
- evolution of flatness:

$$|\Omega_{\rm total} - 1| \propto \begin{cases} t & \text{radiation-dominated era} \\ t^{2/3} & \text{early matter-dominated era} \end{cases} \tag{72}$$

  tiny deviations of $\Omega_{\rm total}$ from unity grow rapidly! *flatness problem*

- where do structures originate from in the first place?

Horizon Problem



Inflation: Idea

- universe can be driven towards flatness by accelerated expansion, $\ddot{a} > 0$; this seems incompatible with gravity
- Friedmann's equation allows accelerated expansion if

$$\rho c^2 + 3p < 0 \,, \quad p < -\frac{\rho c^2}{3} \tag{73}$$

- simple scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{74}$$

has negative pressure if

$$\dot{\phi}^2 < V(\phi) \tag{75}$$

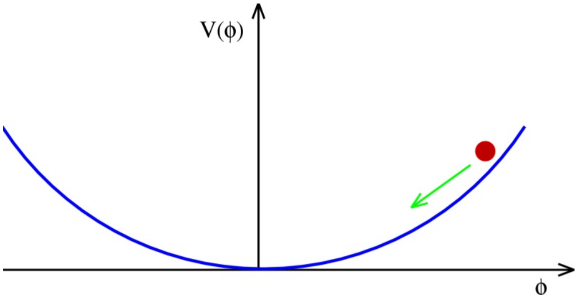
Inflation: Slow Roll, Conditions

- successful inflation under slow-roll conditions:

$$\epsilon \equiv \frac{1}{24\pi G}\left(\frac{V'}{V}\right)^2 \ll 1 \,, \quad \eta \equiv \frac{1}{8\pi G}\left(\frac{V''}{V}\right) \ll 1 \qquad (76)$$

- for solving the flatness problem, increase in the scale factor by a factor of approximately e^{60} is necessary; this would at the same time solve the horizon (or causality) problem
- it is assumed that the inflaton field can decay through some coupling to “ordinary” matter: *reheating*

Slow Roll



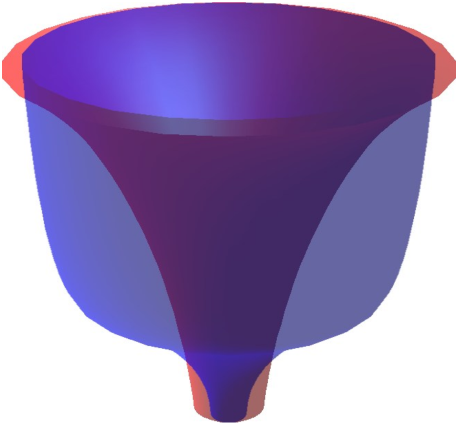
Inflation: Structure Formation

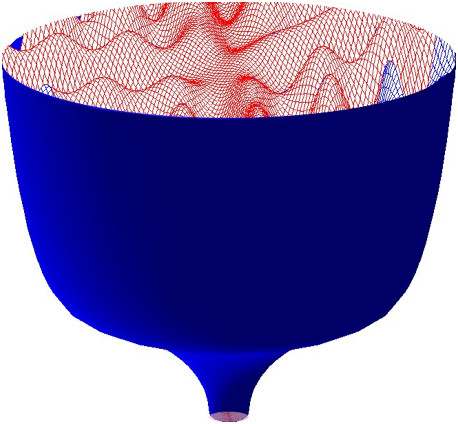
- inflaton field must have undergone vacuum fluctuations
- once inflation sets in, they are quickly driven outside of the horizon, where they "freeze in" because they lack causal contact
- the (primordial) density power spectrum predicted by inflation is

$$P_\mathrm{i}(k) \propto k^n \;, \quad n \lesssim 1 \tag{77}$$

- density fluctuations are expected to be Gaussian because of the central limit theorem
- inflation provides a possibility for solving the horizon and flatness problems and provides a natural explanation for the origin of structures in the universe

Causality and Structure Formation





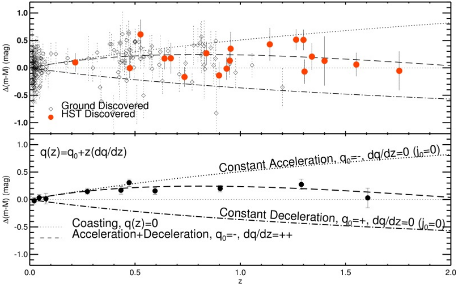
Dark Energy

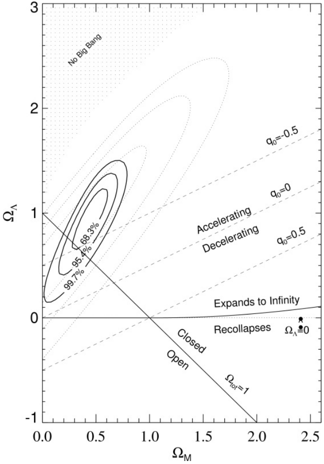
1. Expansion of the Universe
2. Modified Equation of State
3. Effects on Cosmology

Expansion of the Universe

- CMB measurements: universe is spatially flat, i.e. its total energy density equals the critical density
- dark and baryonic matter density, dark and baryonic contributes approximately 30% to the total energy density; abundance of light elements requires the baryon density to be much lower
- supernovae of type Ia: fixed amount of “explosives” blows up; they form a class of “standard candles”
- comparing absolute luminosity and apparent brightness, luminosity distance as a function of redshift can be infered
- high-redshift supernovae show transition from decelaration to acceleration near $z \sim 1$

Type-Ia Supernovae





Modified Equation of State

- cosmological constant is physically dissatisfactory; as for inflation, assume scalar field (“cosmon”, “quintessence”) with negative pressure,

$$p = w\rho c^2 \,, \quad w < -\frac{1}{3} \tag{78}$$

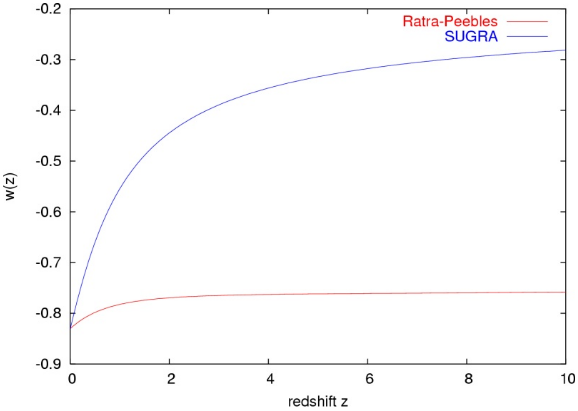
- if w is constant,

$$\rho_{\mathrm{Q}} = \rho_{\mathrm{Q0}} a^{-3(1+w)} \tag{79}$$

- Friedmann equation becomes

$$H^2(a) = H_0^2 \left[\Omega_{\mathrm{m0}} a^{-3} + (1 - \Omega_{\mathrm{m0}} - \Omega_{\mathrm{Q0}}) a^{-2} + \Omega_{\mathrm{Q0}} a^{-3(1+w)}\right] \tag{80}$$

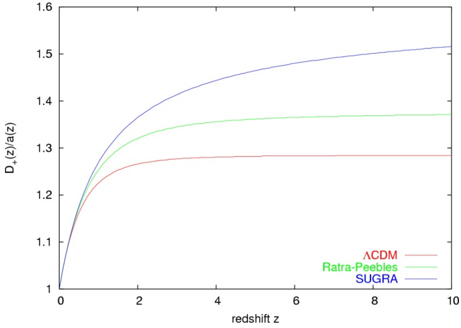
Equation of State

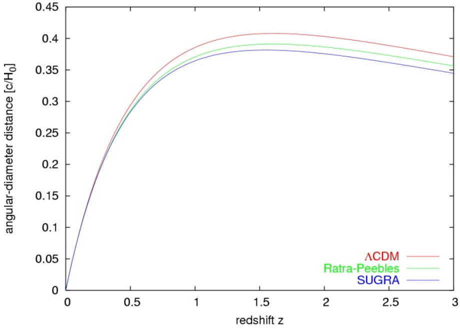


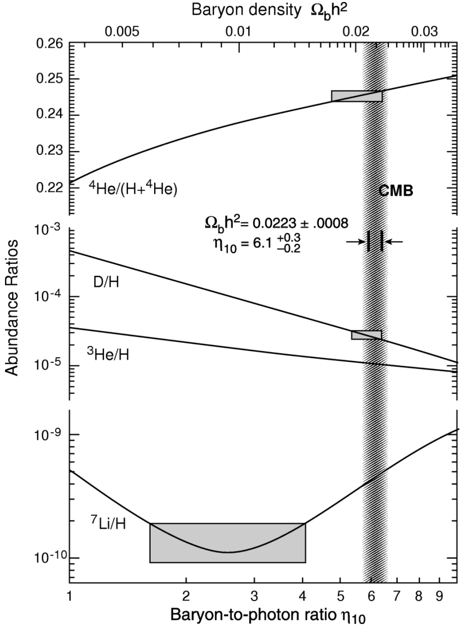
Effects on Cosmology

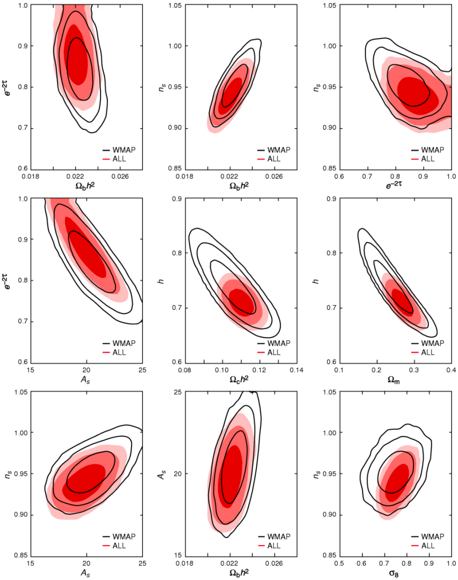
- cosmic expansion during nucleosynthesis is tightly constrained by light-element abundances
- effects on the CMB: width of the recombination shell, amount of Silk damping
- modified angular-diameter and luminosity distances affect supernovae of type Ia, apparent size of CMB fluctuations, cosmic volume, overall geometry of the universe, and gravitational lensing
- the growth factor is modified; structures form earlier in quintessence compared to cosmological-constant models
- dark-matter haloes tend to be denser, which may have strong effects on their appearance

Growth Factor, Distances






Baryon density $\Omega_b h^2$
0.005
0.01
0.02
0.03
0.26
0.25
0.24
0.23
0.22
^{4}He/(H+^{4}He)
CMB
$\Omega_b h^2$= 0.0223 ± .0008
η_{10} = 6.1 $^{+0.3}_{-0.2}$
10^{-3}
10^{-4}
10^{-5}
D/H
^{3}He/H
10^{-9}
10^{-10}
^{7}Li/H
Abundance Ratios
1
2
3
4
5
6
7
8
9
Baryon-to-photon ratio η_{10}


$e^{-2\tau}$
$\Omega_b h^2$
n_s
$\Omega_b h^2$
n_s
$e^{-2\tau}$
$e^{-2\tau}$
A_s
h
$\Omega_c h^2$
h
Ω_m
n_s
A_s
A_s
$\Omega_b h^2$
n_s
σ_8
WMAP
ALL