$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

e.g. $g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2)$$

Ricci Tensor $R_{\mu\nu}$ }
Ricci Skalar R } Functions of $g_{\mu\nu}$

Energy-Momentum Tensor $T_{\mu\nu}$

Function of energy density ρ,
pressure p

Einstein Equations:

cosmological constant ↓

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu} + \Lambda g_{\mu\nu}$$

homogeneous, isotropic

.... → Robertson Walker Metric: $ds^2 = dt^2 - R(t)\left\{ \frac{dr^2}{1-kr^2} + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 \right\}$

r, ϑ, φ : polar coordinates, $R(t)$ scale parameter

k curvature

$k = +1$ closed $\left\{ \begin{array}{l} \bigcirc < 2\pi r \\ \sum_\Delta \measuredangle > 180^\circ \end{array} \right.$

$k = 0$ flat

$k = -1$ open

Friedmann Equations:

(1) $\frac{\dot R^2}{R^2} + \frac{kc^2}{R^2} = \frac{8\pi}{3} G \rho$

(2) $\ddot R = -\frac{4\pi}{3} G R \left(\rho + 3\frac{p}{c^2}\right)$

$R(t),\ \rho(t),\ p(t)$

only 2 equations

→ equation of state:

$$p = w \rho c^2$$

$[FE ①] + FE ②$

$$\Rightarrow \dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0$$

$p = w\rho c^2$ ⇒

$$\dot{\rho} + 3\frac{\dot{R}}{R}\,\rho(1+w)$$

$$\Rightarrow \frac{\dot{\rho}}{\rho} = -3(w+1)\frac{\dot{R}}{R}$$

non relativistic matter ρ_{mat}

$$p = \frac{n}{3} m \langle v^2 \rangle = \frac{n}{3} mc^2 \frac{\langle v^2 \rangle}{c^2} = \frac{\rho_{mat}}{3}\frac{\langle v^2 \rangle}{c^2} \approx 0$$

$$E = \sqrt{p^2c^2 + m^2c^4}$$

$w = 0$ non rel. matter

$\Rightarrow \rho_{mat} \sim \frac{1}{R^3}$

rel. matter / radiation ρ_{rad}

$$p_R = \frac{1}{3}\rho_R c^2 \qquad w = \frac{1}{3}$$

$\Rightarrow \rho_{rad} \sim \frac{1}{R^4}$

$\lambda \sim R$ redshift → 4 instead of 3

ρ_{vac} $\quad w = -1$

$$\Rightarrow \frac{d\rho}{\rho} = -3(w+1)\frac{dR}{R} \Rightarrow \rho \sim R^{-3(w+1)}$$

Hubble parameter: $H = \frac{\dot{R}(t)}{R(t)}$

① $\left(\frac{\dot{R}}{R}\right)^2 + \frac{kc^2}{R^2} = \frac{8\pi}{3} G\rho$

$\Rightarrow \frac{k^2c^2}{H^2R^2} = \frac{8\pi G}{3H^2}\rho - 1$

critical density $\rho_c = \frac{3H^2}{8\pi G} = \frac{5\,\text{GeV}}{m^3}$ $\left[\frac{5\,\text{protons}}{m^3}\right]$

$\Omega = \rho/\rho_c = \Omega_{mat} + \Omega_{rad} + \Omega_{vac}$

$$\frac{kc^2}{H^2R^2} = \Omega - 1$$

$\Omega \; > \; = \; < \; 1$: closed / flat / open

$\Rightarrow$ flat universe

flat universe $\Rightarrow k = 0$

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi}{3} G\rho$$

$\rho_{rad} \sim 1/R^4$

$\rho_{mat} \sim 1/R^3$

$\rho_{vac} = \text{const}$

matter dominated

$\Rightarrow R \sim t^{2/3}$

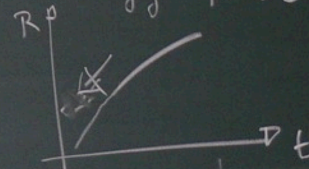
since $R \sim 1/T$ $\Rightarrow H \sim T^{3/2}$

radiation dominated

$\Rightarrow R \sim \sqrt{t}$

$H \sim T^2$

vacuum energy $R \sim e^t$



Universe grows

→ Big Bang

Planck's law $dn(\nu, T)$

$R_1 \to R_2 = R_1 (1+z)$

$dn_2 = \frac{dn_1}{(1+z)^3}$; $\lambda_2 = \lambda_1 (1+z)$

→ remains a Planck distribution

with a new $T_2 = T_1/(1+z)$

$R \sim \frac{1}{T}$

today: ν, γ | e^-, p, n

$n_\gamma / n_p \sim 10^{-9}$

$\gamma + \gamma \leftrightarrow e^+ + e^-$

$kT \approx 5\,MeV$

$\gamma + \gamma \leftrightarrow \mu^+ + \mu^-$

$kT \sim 180\,MeV$

$\leftrightarrow \pi^+ + \pi^-$

rel. particles / radiation $E = pc$

$x = pc/kT$

$\rho(T) \sim (kT)^4 \cdot g \begin{cases} 1 & \text{Bosons} \\ 7/8 & \text{Fermions} \end{cases}$

number density $n(T)$

phase space points

$$\int_0^\infty \frac{4\pi p^2 dp}{h^3} \, g \, \frac{1}{e^{E/kT} \mp 1}$$

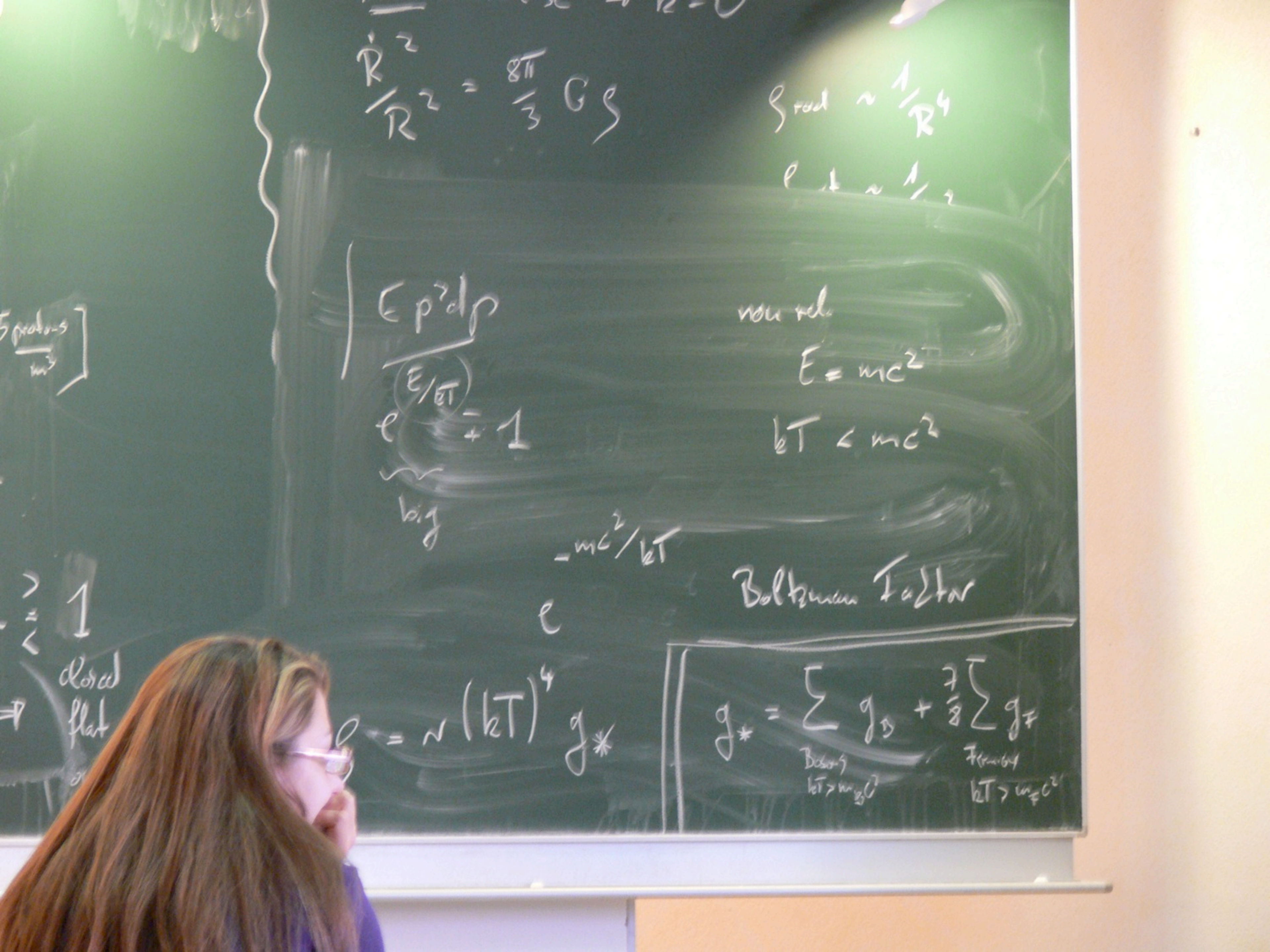
g: Spins, internal degree's

energy density:

$$\rho(T) = \frac{4\pi g}{h^3} \int \frac{E \, p^2 dp}{e^{E/kT} \mp 1}$$

− Bosons
+ Fermions

$E = \sqrt{p^2 c^2 + m^2 c^4}$



Ṙ²/R² = 8π/3 G ρ
ρ rad ~ 1/R⁴
E p² dp / (e^(E/kT) ∓ 1)
big
non rel.
E = mc²
kT < mc²
e^(−mc²/kT)
Boltzmann Faktor
closed
flat
ρ = ~ (kT)⁴ g*
g* = Σ g_B + 7/8 Σ g_F
Bosons
Fermions

1

$= \frac{5\,GeV}{m^3} \quad \left[\frac{5\,protons}{m^3}\right]$

$\Omega_{rad} + \Omega_{vac}$

$-1 \Big/ \quad \Omega \; \substack{> \\ = \\ <} \; 1$

closed

⇒ flat universe

$\frac{E\,p^2\,dp}{e^{E/kT} \mp 1}$

e^{-m}

$(kT)^4 \; a$

$kT \sim 10^{19}$ GeV $\quad t \sim 10^{-43}$ sec

quantum theory of gravity ??

↑ ?? Planck epoch

$kT \sim (10^{15} - 10^{19})$ GeV $\quad t \sim (10^{-43} - 10^{-35})$ sec

electroweak + strong unified

↑ GUT epoch

at $kT \sim 10^{15}$ GeV $\quad t \sim 10^{-35}$ sec

symmetry breaking → strong, → electroweak

→ phase transition

⋮

at $kT < 1$ TeV known physics

$t \sim 10^{-12}$ sec

→ $t\bar{t}$ disappear. $-10\frac{1}{2}$

→ Higgs, $W^\pm Z_0$ → phase transition → weak, → electromagnetic

$\quad -1 \qquad -3$

g_* ↑

quark epoch

$$g_B = \underbrace{3\times 3}_{Z,W^\pm} + \underbrace{2}_{\gamma} + \underbrace{1}_{\text{Higgs}} + \underbrace{8\times 2}_{\text{gluons}} = 28$$

$$g_F = \underbrace{6\times 2}_{e^\pm,\mu^\pm,\tau^\pm} + \underbrace{\underbrace{12}_{q\bar{q}} \times \underbrace{2}_{\text{spin}} \times \underbrace{3}_{\text{color}}}_{72} + 3\times 2 = 90$$

$$g_* = 28 + \frac{7}{8}\cdot 90 = 106\frac{3}{4}$$

63 from quarks

$$\frac{7}{8}\,72 = 63$$

- $b\bar{b}$, $\tau^{\pm}$ — $3\frac{1}{2}$
- $c\bar{c}$, $s\bar{s}$ — $31\frac{1}{2}$ → $g^* = 51\frac{1}{4}$

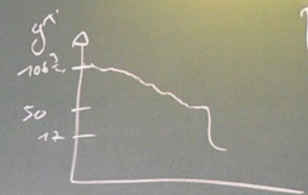
(21 quarks)

⇒ kT ~ 170 MeV QCD phase transition

g, $u\bar{u}$, $d\bar{d}$ —condense→ $\pi^{\pm}$ π^{0} → $17\frac{1}{4}$

16 21 3

quark epoch



$H = \dot{R}/R$

$R \sim 1/T$

·) $\rho_{rad} \sim \frac{1}{R^4}$; $\rho_{mat} \sim \frac{1}{R^3}$

$p \sim \rho$ $p = 0$

·) FEQ flat Universe ($k = 0$)

⇒ $H^2 \sim \rho$

$\rho_{rad} \sim g^* T^4$ } ⇒ $H \sim \sqrt{g^*}\, T^2$ flat, rad. dom.

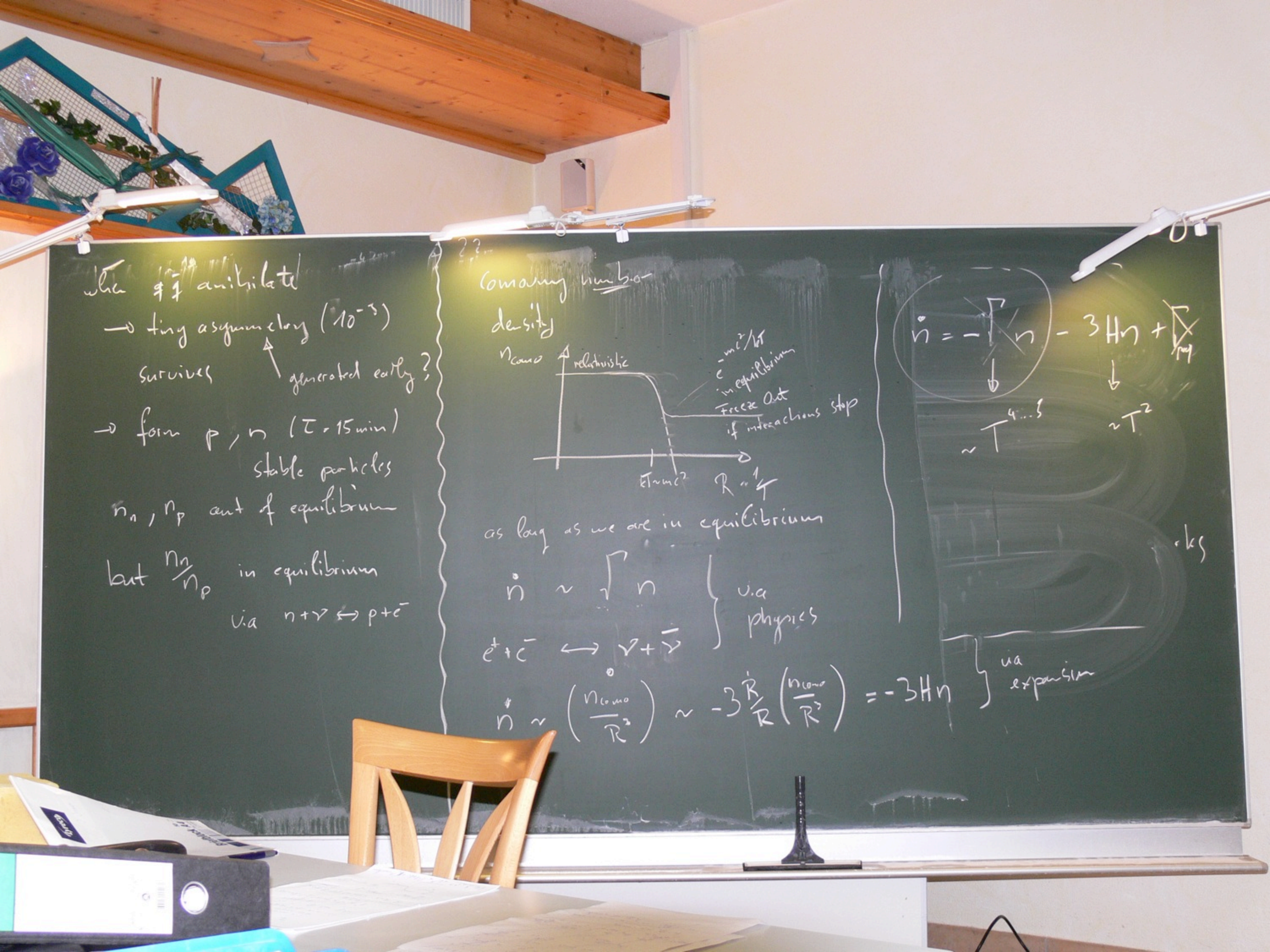
·) $\rho_{rad/relativistic} \sim g T^4$

$\rho_{non\,rel} \sim (mc^2)\, e^{-mc^2/kT}$

↳ non rel. species dont count (⚠ in equilibrium...)

⇒ let's count the rel. degrees of freedom

$$g^* = \sum g_B + \frac{7}{8} \sum g_F$$


when q q̄ annihilate
→ tiny asymmetry (10^-9)
survives
generated early?
→ form p, n (τ = 15 min)
stable particles
n_n, n_p out of equilibrium
but n_n/n_p in equilibrium
via n + ν ↔ p + e^-
comoving number density
n_como
relativistic
e^(-mc²/kT) in equilibrium
Freeze Out if interactions stop
kT ~ mc²
R ~ 1/T
as long as we are in equilibrium
ṅ ~ Γ n
e^+ + e^- ↔ ν + ν̄
via physics
ṅ ~ (n_como / R³)˙ ~ -3 (Ṙ/R) (n_como / R³) = -3Hn
via expansion
ṅ = -Γ n - 3Hn + P_prod
~ T^(4...3)
~ T²

- $b\bar{b}$, $\tau^{\pm}$ — $3\frac{1}{2}$
- $c\bar{c}$, $s\bar{s}$ — $31\frac{1}{2}$ ⟶ $g^* = 51\frac{1}{4}$ (21 quarks)

⇒ $kT \sim 170\,MeV$ QCD phase transition

g, $u\bar{u}$, $d\bar{d}$ ⟶(condense) $\pi^{\pm}$ π^{0} ⟶ $17\frac{1}{4}$

16, 21, 3

·) π disappear, $\mu^{\pm}$

- $kT \sim 2\,MeV$ γ, e^+e^-, $\nu\bar{\nu}$ $t \sim 1\,sec$

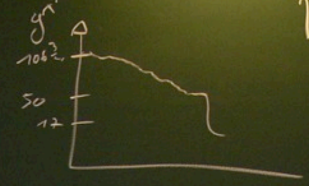
→ ν's freeze out, decouple

since then number of ν's is constant

quark epoch

$H = \dot{R}/R$

$R \sim 1/T$



$n_{\nu,\,today} = 336\ cm^{-3}$

112 per flavour

$$\Omega_\nu = \frac{112 \cdot \sum_{i=1}^{3} m_i}{5\,keV/cm^3} \approx \frac{\sum m_i}{45\,eV}$$

·) $\rho_{rad} \sim \frac{1}{R^4}$; $\rho_{mat} \sim \frac{1}{R^3}$

$p \sim \rho$ $p = 0$

·) FE@ flat Universe ($k=0$)

⇒ $H^2 \sim \rho$

$\rho_{rad} \sim g^* T^4$ } ⇒ $H \sim \sqrt{g^*}\,T^2$ flat, rad. dom.

·) $\rho_{rad/relativistic} \sim g T^4$

$\rho_{non\ rel} \sim (mc^2)\, e^{-mc^2/kT}$

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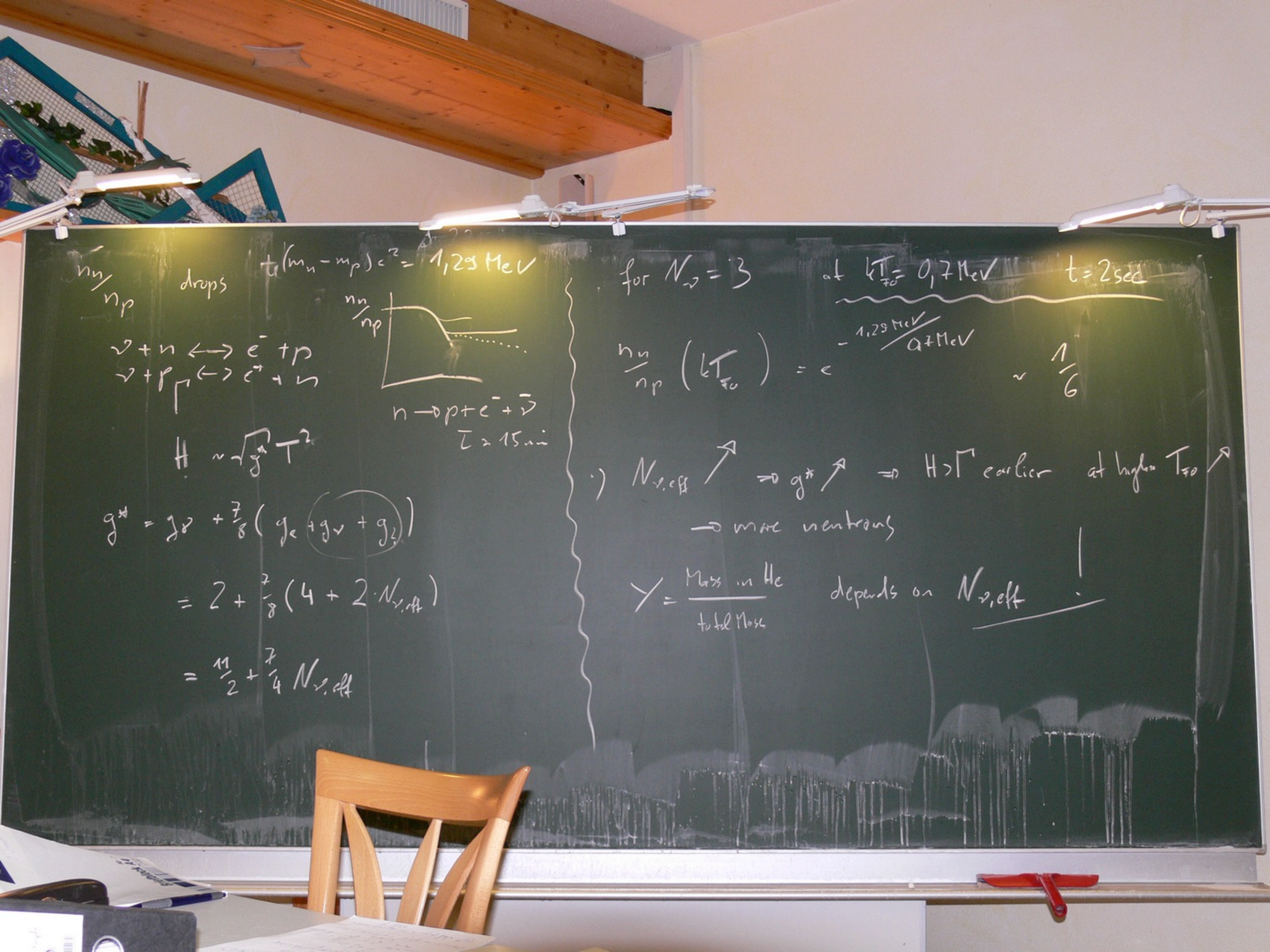
⇒ let's count the rel. degrees of freedom

$$g^* = \sum g_B + \frac{7}{8} \sum g_F$$



n_n/n_p drops
(m_n − m_p) c² = 1,29 MeV
n_n/n_p
ν + n ⟷ e⁻ + p
ν + p ⟷ e⁺ + n
Γ
n → p + e⁻ + ν̄
τ ≈ 15 min
H ~ √g* T²
g* = g_γ + 7/8 (g_e + g_ν + g_τ)
= 2 + 7/8 (4 + 2 · N_ν,eff)
= 11/2 + 7/4 N_ν,eff
for N_ν = 3
at kT_FO = 0,7 MeV
t = 2 sec
n_n/n_p (kT_FO) = e^(− 1,29 MeV / 0,7 MeV) ~ 1/6
N_ν,eff ↗ ⇒ g* ↗ ⇒ H > Γ earlier at higher T_FO ↗
→ more neutrons
Y = Mass in He / total Mass
depends on N_ν,eff !

$kT \sim 0{,}5\,\mathrm{MeV}$ $\quad t \sim 10\,\mathrm{sec}$

$e^+ + e^- \leftrightarrow \gamma + \gamma$

$e^\pm$ disappear $\Rightarrow$ additional γ

$\Rightarrow T_{CMB} > T_{C\nu B}$

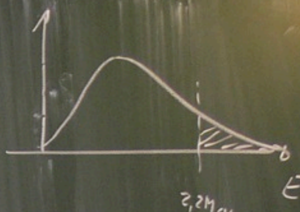
2,7 K $\quad$ 1,9 K

$n_{Planck}(2{,}7\,K) = 411\ cm^{-3}$

$n\ (1{,}9\,K) = 336\ cm^{-3}$

n decay as long "$\leftarrow$" 2,2 MeV

until $n + p \rightleftarrows d + \gamma$



as long as $n_\gamma(E > 2{,}2\,\mathrm{MeV}) \gtrsim n_B$

$\eta = \frac{n_{bar}}{n_\gamma} = 4 \cdot 10^{-10}$

"$\leftarrow$"

$\Rightarrow 10^9 \times$ more γ

only at $kT \sim 0{,}1\,\mathrm{MeV}$ "$\leftarrow$" stops

$t \sim 3\,\mathrm{min} \rightarrow d \rightarrow He$ is formed

$(d, {}^3He, {}^7Li, He, H)$

$e^{-3/15} \approx 4/5$

all n go into He

$\Rightarrow \frac{n_n}{n_p} = \frac{1}{6} \cdot \frac{4}{5} = \frac{1}{7}$

n_n/n_p drops $t_i (m_n - m_p)c^2 = 1{,}29$ MeV

$\nu + n \longleftrightarrow e^- + p$

$\bar{\nu} + p \longleftrightarrow e^+ + n$

Γ

$n \to p + e^- + \bar{\nu}$

$\tau \approx 15$ min

$H \sim \sqrt{g^*}\, T^2$

$g^* = g_\gamma + \frac{7}{8}\left(g_e + g_\nu + g_? \right)$

$= 2 + \frac{7}{8}\left(4 + 2 \cdot N_{\nu,eff} \right)$

$= \frac{11}{2} + \frac{7}{4} N_{\nu,eff}$

$N_{He} = \frac{N_n}{2}$

$N_H = N_p - N_n$

for $N_\nu = 3$ at $kT_{FO} = 0{,}7$ MeV $t = 2$ sec

$\frac{n_n}{n_p}\left(kT_{FO} \right) = e^{-\frac{1{,}29\,\text{MeV}}{0{,}7\,\text{MeV}}} \sim \frac{1}{6}$

·) $N_{\nu,eff} \nearrow \;\Rightarrow g^* \nearrow \;\Rightarrow H > \Gamma$ earlier at higher $T_{FO} \nearrow$

→ more neutrons

$Y = \frac{\text{Mass in He}}{\text{total Mass}}$ depends on $N_{\nu,eff}$!

$= \frac{4 \cdot N_{He}}{4 N_{He} + N_H} = \frac{2\, n_n/n_p}{n_n/n_p + 1} = \frac{1}{4}$

$\rightarrow T_{CMB} > T_{C\nu B}$

2,7 K   1,9 K

$n_{Planck}(2{,}7\,K) = 411\ cm^{-3}$

$n_{\nu}(1{,}9\,K) = 336\ cm^{-3}$

measure $Y \xrightarrow{\text{get}} \eta$

$\rightarrow \eta \nearrow \rightarrow$ less $\gamma \rightarrow$ "$\leftarrow$" stops earlier

$\rightarrow$ more neutrons $\rightarrow Y \nearrow$

$\eta \Rightarrow n_B \sim 0{,}2\ m^{-3}$

$n_\gamma = 411\ cm^{-3}$

$\Rightarrow \Omega_B \approx 0{,}04$) DM

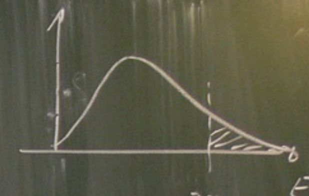
$\Omega_{matter} = 0{,}3$

to get η, $N_{\nu,eff}$

need Y ; $\tau_n = 15\ min$

n decay as long "$\leftarrow$" 2,2 MeV

until $n + p \rightleftarrows d + \gamma$



as long as $n_\gamma(E > 2{,}2\,MeV) \gtrsim n_d$

"$\leftarrow$"

$\eta = \frac{n_{bar}}{n_\gamma} = 4 \cdot 10^{-10}$

$\Rightarrow 10^9 \times$ more γ

only at $kT \sim 0{,}1\ MeV$ "$\leftarrow$" stops

$t \sim 3\,min \rightarrow d \rightarrow He$ is formed

$(d, {}^3He, {}^7Li, He, H)$

$e^{-3/15} \approx 4/5$

all n go into He

$\Rightarrow \frac{n_n}{n_p} = \frac{1}{6} \cdot \frac{4}{5} = \frac{1}{7}$

$\Omega_{matter} \sim 7 \cdot \Omega_B$

$$\frac{\Omega_{matter}}{\Omega_\gamma} = \frac{7 \cdot \Omega_B}{\Omega_\gamma} = 7 \cdot \frac{1\,GeV}{kT} \cdot \frac{n_{Bar}}{n_\gamma}$$

$$= \frac{1\,GeV \; 6 \cdot 10^{-10}}{kT} \approx 3eV/kT$$

at $kT > 3eV$ ⇒ Universe is radiation dominated

$< 3eV$ ⇒ matter dominated !

-) $kT \sim eV$ $t \sim 10\,000$ years

matter - radiation equality

structure formation can start !

(non electromagnetic matter)

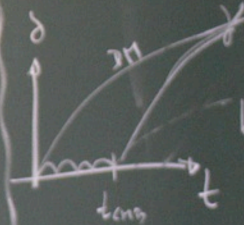
$p + e^- \rightleftarrows H + \gamma_{13,6\,eV}$

$T = 3000\,K$

$T_{today} = 2{,}7\,K$

$kT \sim 0{,}3\,eV$ $t \sim 360\,000$ years

$z = 1100$



decouple ⇒ CMB forms

baryonic matter can form structures

t (electromagnetic interacting matter)

tomorrow : $\frac{\delta\rho}{\rho} = \delta$ $\delta(t) \sim R(t)$

density contrast

⇒ needs $\frac{\delta\rho}{\rho} \sim 10^{-3}$ at CMB

but 10^{-5} !

Structure Formation

$\rho \to \frac{\delta\rho}{\rho} = \delta(x,t)$ Density contrast

p pressure

$\vec{v}$ velocity

ϕ gravity

(3) $\vec\nabla^2\phi = 4\pi G\rho$

$\phi = \phi_0 + \phi_1$

(1) $\dot\rho + \vec\nabla(\rho\vec v) = 0$

(2) $\underbrace{\frac{\partial \vec v}{\partial t} + (\vec v\vec\nabla)\vec v}_{\frac{d\vec v}{dt} = \vec a} + \overbrace{\frac{1}{\rho}\vec\nabla p}^{\frac{v_s^2}{\rho}\vec\nabla\rho} + \vec\nabla\phi = 0$

$\partial p/\partial\rho = v_s^2$; $p = w\rho c^2$

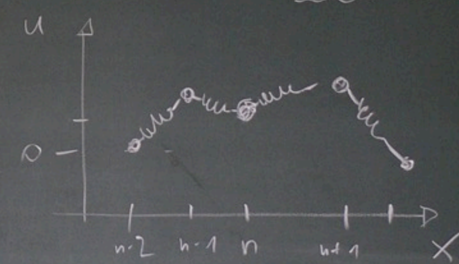
$w = v_s^2/c^2$

linearize: $\rho = \rho_0 + \rho_1$ $\quad \rho_1 \ll \rho_0$ $\quad \delta = \rho_1/\rho_0$

$\vec v = \vec v_0 + \vec v_1$; $p = p_0 + p_1$

x_0 fullfill equations ; $v_0 = 0$; $x_1 y_1$ neglect

$\dot{(1)}$ use (2) $\Rightarrow$ $\frac{\partial^2\delta}{\partial t^2} - v_s^2\vec\nabla^2\delta = 4\pi G\rho_0\delta$ *



$m\ddot u_n = D(u_{n+1} - u_n) - D(u_n - u_{n-1}) = D\,\partial^2 u/\partial x^2$

$v_s^2 = D/m$

$\Rightarrow \frac{\partial^2 u}{\partial t^2} - v_s^2\frac{\partial^2 u}{\partial x^2} = 0$

$$\delta = \sum_k \delta_k(t)\, e^{i\vec{k}\vec{r}}$$

$$\overset{*}{\Rightarrow} \frac{\partial^2 \delta_k}{\partial t^2} + \left(v_s^2 k^2 - 4\pi G \rho_0\right)\delta = 0$$

$$\delta_k(t) = \delta_{k\omega}\, e^{i\omega(k)t}$$

$$\Rightarrow \omega^2 = v_s^2 k^2 - 4\pi G \rho_0$$

ω real $\rightarrow$ oscillating modes $\omega^2 > 0$

ω imaginary $\omega = i|\omega| \rightarrow$ growing (decaying) modes $\omega^2 < 0$

$$e^{i\omega t} \rightarrow e^{\pm|\omega|t}$$

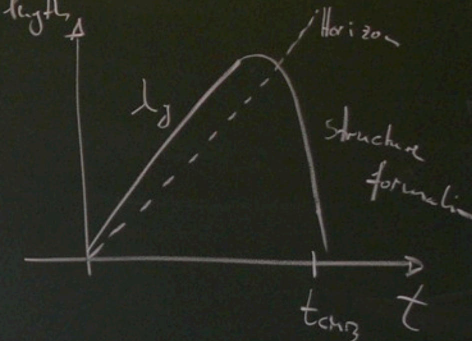
structure growth only if $k < \sqrt{\frac{4\pi G \rho_0}{v_s^2}}$

$k = 2\pi/\lambda \Rightarrow \lambda_J > \sqrt{\frac{\pi v_s^2}{G \rho_0}}$ Jeans length

$\lambda_J \overset{!}{<}$ Horizon $c \cdot t$ otherwise no structure growth at all

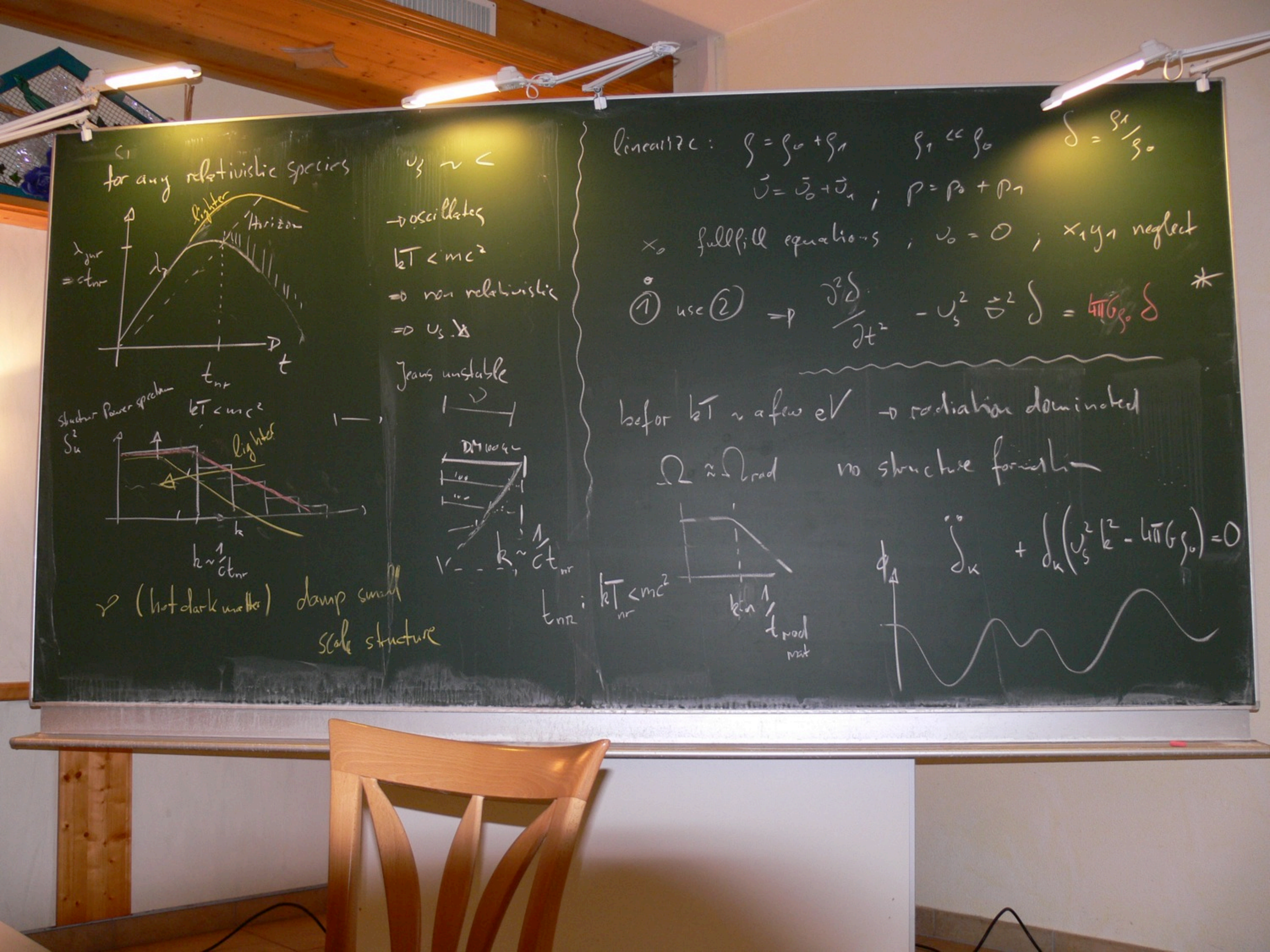
$\lambda_J \sim v_s / \sqrt{\rho} \sim v_s \sqrt{R^3} \sim v_s t$

mat. dominated $R \sim t^{2/3}$



Early plasma, "before CMB decoupling"

$\Rightarrow$ no "baryonic" structure formation before t_{CMB}



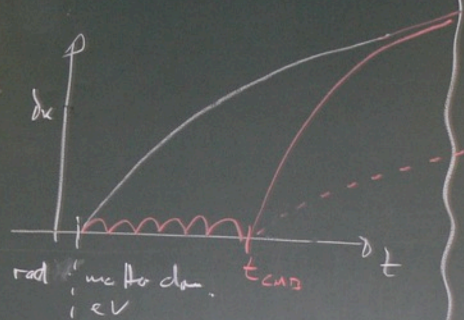
for any relativistic species
lighter
Horizon
t
t_{nr}
→ oscillates
$kT < mc^2$
⇒ non relativistic
Jeans unstable
structure Power spectrum
$kT < mc^2$
lighter
$k \sim \frac{1}{ct_{nr}}$
$k \sim \frac{1}{ct_{nr}}$
ν (hot dark matter) damp small scale structure
t_{nr} : $kT_{nr} < mc^2$
linearize: $\rho = \rho_0 + \rho_1$ $\rho_1 \ll \rho_0$ $\delta = \rho_1 / \rho_0$
$\vec{v} = \vec{v}_0 + \vec{v}_1$; $p = p_0 + p_1$
x_0 fullfill equations ; $v_0 = 0$; $x_1 y_1$ neglect
① use ② ⇒ $\frac{\partial^2 \delta}{\partial t^2} - v_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta$
befor kT ~ a few eV → radiation dominated
$\Omega \approx \Omega_{rad}$ no structure formation
$\ddot{\delta}_k + \delta_k (v_s^2 k^2 - 4\pi G \rho_0) = 0$
t_{rad}
mat

expanding Universe:

$\rho \to \rho / R^3$; $k \to k / R$

$\vec{v}_0 = 0 \to \frac{\dot{R}}{R} \vec{r}$

use $\delta = \sum \delta_k e^{ikr}$



$$\ddot{\delta}_k + 2 \frac{\dot{R}}{R} \dot{\delta}_k + \left(v_s^2 \frac{k^2}{R^2} - 4\pi G \rho_0 \right) \delta_k = 0$$

growing modes $p = 0 \Rightarrow v^2 k^2 \approx 0$

$$\ddot{\delta}_k + 2 \frac{\dot{R}}{R} \dot{\delta}_k = 4\pi G \rho_0 \delta_k$$

$$\ddot{\delta}_k + \frac{4}{3t} \dot{\delta}_k - \frac{2}{3t^2} \delta_k = 0$$

matter dom: $R \sim t^{2/3} \Rightarrow H = \frac{\dot{R}}{R} = \frac{2}{3t}$

$$4\pi G \rho_0 \underset{\text{FE ① flat}}{=} \frac{3}{2} H^2 = \frac{2}{3t^2}$$

$\delta_k \sim t^n$ $\quad n(n-1) + \frac{4}{3} n - \frac{2}{3} = 0$

$$\to n = \begin{cases} 2/3 & \Leftarrow \text{growing} \\ -1 & \text{decaying} \end{cases}$$

$z_{CMB} = 1000$

$$\delta_k(t) \sim t^{2/3} \sim R$$

$\frac{R_{today}}{R_{CMB}} = 1000$

δ_k grows like R !